Chapter 4

Oscillations and resonance

4.1 Modeling oscillations

Exercise 4.1.1. *Consider a generic oscillator equation*

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} = ky = 0.$$

Suppose that $y_1(t)$ and $y_2(t)$ are solutions. Explain why it follows $\alpha y_1(t) + \beta y_2(t)$ is a solution for any choice of constants α and β. This is called the ***superposition principle****.*

Exercise 4.1.2. *Consider the second order differential equation*

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

1. *Find the general solution to this differential equation.*
2. *Solve the initial value problem*

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

Exercise 4.1.3. *Find the general solution of the following equations:*

1. $\frac{d^2y}{dt^2} + \omega^2 y = 0$;
2. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$;
3. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = 0$;

4. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 0$;

5. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$.

Exercise 4.1.4. *Consider the differential equation*

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0.$$

1. *Explain why the superposition principle applies to this equation. That is, show that that if $y_1(t)$ and $y_2(t)$ are solutions then so is $\alpha y_1(t) + \beta y_2(t)$ for any constants α and β.*

2. *Find those values of α for which the function $y(t) = t^\alpha$ solves the differential equation.*

3. *Use the superposition principle to solve the IVP:*

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0, \qquad y(1) = 2, \quad y'(1) = 4.$$

4.2 Oscillators with forcing

In the previous sections we developed a good understanding of solutions to homogeneous equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{4.2.1}$$

Our goal now is to understand what happens when we introduce forcing. Thus we study equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f, \tag{4.2.2}$$

where f is some function of t, and where a, b, c are constants.

In order to handle forcing, we introduce the following "generalized superposition principle."

Generalized superposition principle Suppose $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous equation (4.2.1) and that $y_p(t)$ is a solution to the inhomogeneous equation (4.2.2). Then

$$y(t) = \alpha y_1(t) + \beta y_2(t) + y_p(t)$$

is a solution to (4.2.2) for any constants α and β.

The generalized superposition principle can be physically interpreted as follows: Suppose we have an inhomogeneous equation of the form (4.2.2). Then the general solution

$$y_h(t) = \alpha y_1(t) + \beta y_2(t) \tag{4.2.3}$$

to the associated homogeneous equation (4.2.1) is the "natural response" of the system, in the absence of any external forcing. The function $y_h(t)$ is often called the ***homogeneous solution***. The function $y_p(t)$ is an additional contribution to $y(t)$ that represents the "response" of the system to the external forcing. The function $y_p(t)$ is usually called a ***particular solution***.

The generalized superposition principle suggests an approach for finding the general solution to equations of the form (4.2.2):

1. Find the general solution $y_h(t)$ to the homogeneous equation,
2. Find a particular solution $y_p(t)$ to the inhomogeneous equation,
3. Construct the general solution $y(t) = y_p(t) + y_h(t)$ to the inhomogeneous equation.

Since homogeneous equations are well understood, the general superposition principle means that if we can find one solution to an inhomogeneous equation, then we can easily find them all!

However, we still have to find one solution to (4.2.2). Unfortunately, the most efficient method that mathematicians have been able to come up with thus far is... educated guess and check. The guiding principle is this:

> *Look for a particular solution $y_p(t)$ that is the same "type" of function that $f(t)$ is.*

If you find this unsatisfying, you have good company. There is general theory out there about constructing particular solutions. But the educated guess-and-check method is so much more efficient for the simple cases we'll cover in this class that it simply isn't worth it to get out the fancy theory at this point.

The following list of examples demonstrates how the educated guess and check method works.

Example 4.2.1. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 7.$$

First we consider the associated homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation is $\lambda^2 + 4 = 0$, and thus the eigenvalues are $\lambda = \pm 2i$. This implies that the homogeneous solution is

$$y_h(t) = \alpha \cos(2t) + \beta \sin(2t).$$

We now try to guess that a particular solution. Since the forcing function f is constant, we guess that the particular solution takes the form $y_p(t) = a$ for some constant a. Plugging this in to the original equation yields

$$0 + 4a = 7.$$

Thus we obtain a solution $y_p(t) = 7/4$.

Assembling the pieces, we see that the general solution is

$$y(t) = \alpha \cos(2t) + \beta \sin(2t) + \frac{7}{4}.$$

Example 4.2.2. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 7t.$$

First we consider the associated homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation is $\lambda^2 + 4 = 0$*, and thus the eigenvalues are* $\lambda = \pm 2i$*. This implies that the homogeneous solution is*

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We now try to guess that a particular solution. Since the forcing function f *is a linear function, we guess that the particular solution takes the form* $y_p(t) = a + bt$ *for some constants* a, b*. Plugging this in to the original equation yields*

$$0 + 4(a + bt) = 7t.$$

We rearrange this to

$$4a + (4b - 7)t = 0.$$

In this last equation, we have equality in the sense of functions. Thus we must have $a = 0$ *and* $b = 7/4$*. Thus the particular solution is*

$$y_p(t) = \frac{7}{4}t$$

and the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) + \frac{7}{4}t.$$

Example 4.2.3. *Let's analyze the solutions to the equation*

$$\frac{d^2y}{dt^2} + 4y = e^{2t}.$$

First we address the homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

From the previous examples, we know that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We now look for a particular solution. Since the forcing function is exponential with growth rate 2*, we look for a particular solution of the form*

$$y_p(t) = ae^{2t}$$

where a *is some constant. Plugging this in to the differential equation yields*

$$4ae^{2t} + 4ae^{2t} = e^{2t}.$$

It is easy to see that choosing $a = 1/8$ leads to a particular solution

$$y_p(t) = \frac{1}{8}e^{2t}$$

and thus the general solution is

$$y(t) = \alpha \cos(2t) + \beta \sin(2t) + \frac{1}{8}e^{2t}.$$

Notice that as $t \to \infty$, we have $y_p(t) \gg y_h(t)$ and thus the forced response dominates behavior far in the future. As $t \to -\infty$ we have $y_p(t) \ll y_h(t)$ and thus the natural response dominates far in the past.

Activity 4.2.1. *Find the general solution to the differential equation*

$$\frac{d^2y}{dt^2} + 4y = e^{-2t}.$$

In what regime is y_h dominant? In what regime is y_p dominant?

Finally, we give an example in which we solve an initial value problem.

Example 4.2.4. *Let's solve the initial value problem*

$$\frac{d^2y}{dt^2} + 16y = 2t + 1, \qquad y(0) = 2, \quad y'(0) = 0.$$

To accomplish this, we first find the general solution.

The homogeneous equation is

$$\frac{d^2y}{dt^2} + 16y = 0,$$

which has characteristic equation $\lambda^2 + 16 = 0$. Thus the eigenvalues are $\lambda = \pm 4i$ and the homogeneous solution is

$$y_h(t) = \alpha \cos(4t) + \beta \sin(4t).$$

Since the forcing function is linear, we look for a particular solution of the form $y_p(t) = a + bt$. Plugging this in to the original equation yields

$$16(a + bt) = 2t + 1.$$

This is satisfied if we choose $a = 1/16$ and $b = 1/8$. Thus the particular solution is

$$y_p(t) = \frac{1}{16} + \frac{1}{8}t$$

and the general solution is

$$y(t) = \alpha \cos(4t) + \beta \sin(4t) + \frac{1}{16} + \frac{1}{8}t.$$

We now impose the initial conditions. The condition that $y(0) = 2$ becomes

$$2 = \alpha + \frac{1}{16}.$$

Computing

$$y'(t) = -4\alpha \sin(4t) + 4\beta \cos(4t) + \frac{1}{8}$$

we see that the condition $y'(0) = 0$ becomes

$$0 = 4\beta + \frac{1}{8}.$$

Thus $\alpha = 31/16$ and $\beta = -1/32$, which implies that the solution to the initial value problem is

$$y(t) = \frac{31}{16}\cos(4t) - \frac{1}{32}\sin(4t) + \frac{1}{16} + \frac{1}{8}t.$$

Exercise 4.2.1. *Consider the non-homogeneous equation:*

$$\frac{d^2y}{dt^2} + 9y = 9.$$

This equation arises from studying a frictionless oscillator with constant forcing.

1. *Find a particular solution of this equation.*
2. *Find the homogeneous solution.*
3. *Based on the above find the general solution of the equation.*
4. *Solve the IVP*
$$\frac{d^2y}{dt^2} + 9y = 9, \qquad y(0) = 0, \qquad \frac{dy}{dt}(0) = 3.$$
5. *Graph the solution of the IVP in the ty-plane, paying particular attention to long-term behavior of the graph.*
6. *How would you in words describe the effect the forcing has on the oscillator?*

Exercise 4.2.2. *Repeat Problem* **??** *for the differential equation $\frac{d^2y}{dt^2} + 9y = 10e^{-t}$ and IVP*

$$\begin{cases} \frac{d^2y}{dt^2} + 9y = 10e^{-t}, \\ y(0) = 0, \quad \frac{dy}{dt}(0) = -7. \end{cases}$$

Exercise 4.2.3. *Repeat Problem **??** for the equation* $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1$ *and the IVP*

$$\begin{cases} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1, \\ y(0) = 0, \quad \frac{dy}{dt}(0) = 0. \end{cases}$$

Note: this equation represents an oscillator with friction.

Exercise 4.2.4. *Repeat Problem **??** for the equation* $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1 + 3e^t$ *and the IVP*

$$\begin{cases} \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1 + 3e^t, \\ y(0) = 2, \quad \frac{dy}{dt}(0) = 1. \end{cases}$$

Exercise 4.2.5. *Solve the initial value problem*

$$\begin{cases} \dfrac{d^2y}{dt^2} + 16y = e^{-\frac{t}{10}} \\ y(0) = \dfrac{110}{1601}, \quad y'(0) = \dfrac{-10}{1601} \end{cases}$$

Plot the solution and describe its characteristic features.

Exercise 4.2.6. *Find general solutions of the following differential equations.*

1. $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 6y = 4e^{-2t}$;

2. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + y = 1 + e^{-t}$.

Exercise 4.2.7. *Find the general solution of the equation*

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 2t + 1.$$

Exercise 4.2.8. *Devise a recipe for finding a forced response of the oscillator with polynomial forcing* $f(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t + a_0$.

4.3 Oscillators with trigonometric forcing

In this section we address oscillators with trigonometric forcing. If there is no damping present in the oscillator, then we can proceed essentially as in the previous section.

Example 4.3.1. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(3t).$$

We easily see that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

Since the forcing function is a cosine function with frequency 3*, we look for a particular solution of the form* $y_p(t) = a\cos(3t)$*. Plugging this in to the equation yields*

$$-9a\cos(3t) + 4a\cos(3t) = \cos(3t).$$

From this we deduce that $a = -1/5$ *and thus we have*

$$y_p(t) = -\frac{1}{5}\cos(3t).$$

The general solution is therefore

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{5}\cos(3t).$$

In order to understand this solution, it is helpful to make use of the trigonometric identity

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

to write

$$\cos(3t) = \cos(2t)\cos(t) - \sin(2t)\sin(t).$$

Thus the general solution can be written

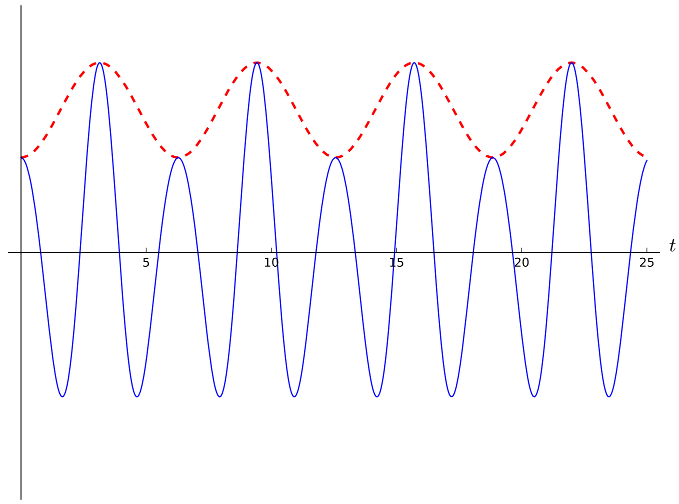
$$y(t) = \left(\alpha - \frac{1}{5}\cos(t)\right)\cos(2t) + \left(\beta + \frac{1}{5}\sin(t)\right)\sin(2t).$$

We interpret the function

$$\left(\alpha - \frac{1}{5}\cos(t)\right)\cos(2t)$$

as the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{5}\cos(t)$*. Thus we see that the solutions* $y(t)$ *oscillate at the frequency associated to the homogeneous equation, but that the amplitudes of these oscillations themselves oscillate at the frequency that is the difference between the forcing frequency and the homogeneous frequency.*

A typical solution looks something like the following:



Here the red dashed line is the graph of the function $\alpha - \frac{1}{5}\cos(t)$*, and the blue solid line is the graph of the function* $(\alpha - \frac{1}{5}\cos(t))\cos(2t)$*.*

The previous example illustrates some interesting interplay between the frequency at which the homogeneous solution oscillates and the frequency of the forcing. In order to discuss this interplay, we introduce some terminology. Let ω_h be the frequency at which solutions to the homogeneous equation oscillate; we call ω_h the ***natural*** frequency. Furthermore, let ω_f be the frequency present in the forcing function; we call ω_f the ***forcing frequency***. The previous example shows that we can expect solutions to a forced oscillator with trigonometric forcing to oscillate at frequency ω_h, and that we can expect the amplitude of these oscillations to themselves oscillate with frequency $\omega_f - \omega_h$.

The following two examples illustrate what happens when the forcing frequency is either very far from, or very close to, the natural frequency.

Example 4.3.2. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(30t).$$

In this case, the forcing frequency $\omega_f = 30$ *is very far from the natural frequency* $\omega_h = 2$*.*

As in the previous example, the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We look for a particular solution of the form $y_p(t) = a\cos(30t)$*. Plugging this in to the equation we obtain*

$$-90a\cos(30t) + 4a\cos(30t) = \cos(30t).$$

Thus we choose $a = -1/86$ *and obtain the particular solution*

$$y_p(t) = -\frac{1}{86}\cos(30t).$$

Consequently, the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{86}\cos(30t).$$

Using the trigonometric identity

$$\cos(30t) = \cos(28t)\cos(2t) - \sin(28t)\sin(2t)$$

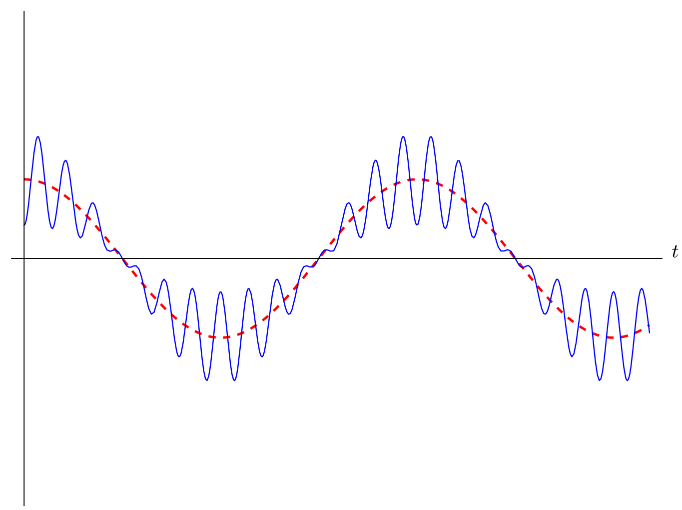
we express the general solution as

$$y(t) = \left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t) + \left(\beta + \frac{1}{86}\sin(28t)\right)\sin(2t)$$

We can interpret the function

$$\left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t)$$

to be the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{86}\cos(28t)$. *Notice that the frequency at which the amplitude is changing is much faster than the natural frequency of the oscillator. Thus a typical solution looks something like the following:*



Here the graph of the function

$$\left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t)$$

is shown with a solid blue curve, while the graph of the function $\alpha\cos(2t)$ is shown with a red dashed curve

Example 4.3.3. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(2.1t).$$

In this case, the forcing frequency $\omega_f = 2.1$ is very close to the natural frequency $\omega_h = 2$.

As in the previous example, the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We look for a particular solution of the form $y_p(t) = a\cos(2.1t)$. Plugging this in to the equation we obtain

$$-4.41a\cos(2.1t) + 4a\cos(2.1t) = \cos(2.1t).$$

Thus we choose $a = -1/86$ and obtain the particular solution

$$y_p(t) = -\frac{1}{0.41}\cos(2.1t).$$

Consequently, the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{0.41}\cos(2.1t).$$

Using the trigonometric identity

$$\cos(2.1t) = \cos(0.1t)\cos(2t) - \sin(0.1t)\sin(2t)$$

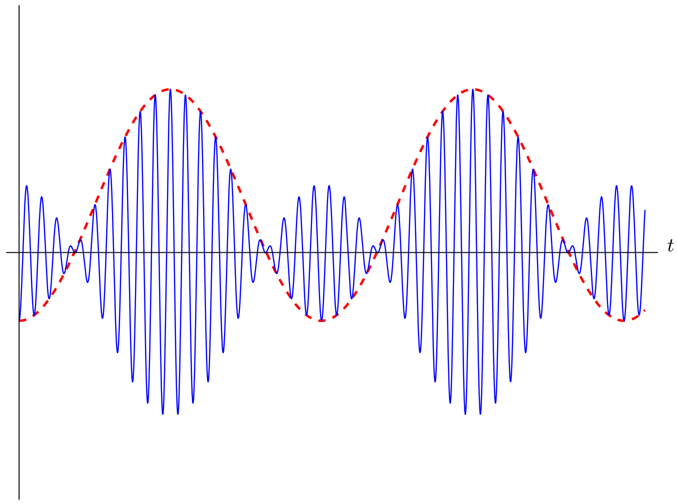
we express the general solution as

$$y(t) = \left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t) + \left(\beta + \frac{1}{0.41}\sin(0.1t)\right)\sin(2t)$$

We can understand the function

$$\left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t)$$

as the function $\cos(2t)$ with amplitude given by $\alpha - \frac{1}{0.41}\cos(0.1t)$. Notice that the frequency changes of the amplitude is much smaller than the frequency of the natural oscillations. This gives rise to solutions that look like the following:



Here the graph of the function

$$\left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t)$$

is shown as a solid blue curve, while the graph of the amplitude function $\alpha - \frac{1}{0.41}\cos(0.1t)$ *is shown as a dashed red curve. The clusters of oscillations shown in this graph are known as* ***beats****, and are typical of cases when the forcing frequency is very close to the natural frequency.*

Beats are frequently used by musicians to tune stringed instruments: Two strings being "in tune" mean that they oscillate at the same natural frequency. If the two strings are very slightly out of tune, then playing one of the strings will have the effect of forcing the other string at a frequency nearby to its natural frequency, resulting in the formation of beats in the vibrations of the second string.

In all of the previous examples, we were able to find a particular solution that was a multiple of the forcing function. Unfortunately, as the following example illustrates, that does not work when there is damping present. However, we are still able to guess a particular solution by considering both cosine and sine functions.

Example 4.3.4. *Consider the equation*

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 4y = \cos(3t),$$

which describes an underdamped oscillator with periodic forcing.

The characteristic equation for the homogeneous equation is

$$\lambda^2 + \lambda + 4 = 0,$$

which has solutions

$$\lambda_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}\, i.$$

Thus the homogeneous solution is

$$y_h(t) = \alpha e^{-t/2} \cos\left(\frac{\sqrt{15}}{2}\, t\right) + \beta e^{-t/2} \sin\left(\frac{\sqrt{15}}{2}\, t\right).$$

In order to find a particular solution, we guess that $y_p(t) = a\cos(3t) + b\sin(3t)$. *Plugging this in to the original equation gives us*

$$\{-9a + 3b + 4a\}\cos(3t) + \{-9b - 3a + 4b\}\sin(3t) = \cos(3t).$$

Thus in order to obtain a solution we need

$$-5a + 3b = 1 \qquad \textit{and} \qquad -3a - 5b = 0.$$

Thus we need $a = -5/34$ *and* $b = 3/34$. *The resulting particular solution is*

$$y_p(t) = -\frac{5}{34}\cos(3t) + \frac{3}{34}\sin(3t).$$

Notice that when guessing our particular solution, we cannot have $b = 0$. *This means that we could not have constructed a particular solution with only* $\cos(3t)$ *in it. (It is a good exercise to try... what goes wrong?)*

Exercise 4.3.1. *The equation*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t)$$

models frictionless oscillations with periodic forcing.

1. *Find the general solution to the homogeneous equation*

$$\frac{d^2y}{dt^2} + 9y = 0.$$

2. *Solve the (homogeneous) initial value problem*

$$\frac{d^2y}{dt^2} + 9y = 0, \qquad y(0) = 0, \quad y'(0) = 5.$$

3. *Find a particular solution of to the inhomogeneous equation*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t).$$

4. *Find the general solution of the equation*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t).$$

5. *Solve the IVP*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t), \qquad y(0) = 0, \quad y'(0) = 5.$$

6. *Graph the solution of the IVP in the ty-plane, paying particular attention to long-term behavior of the graph.*

7. *Describe (in words!) the effect the forcing has on the oscillator.*

Exercise 4.3.2. *Repeat Exercise 4.3.1 for the initial value problem* **→ change the ICs to make th more interesting**

$$\frac{d^2y}{dt^2} + 16y = 7\sin(3t) \qquad y(0) = 0, \quad y'(0) = 0.$$

Exercise 4.3.3. *Repeat Problem 4.3.1 for the initial value problem* **→ change the ICs to make th more interesting**

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin(t) \qquad y(0) = 0, \quad y'(0) = 0.$$

Exercise 4.3.4. *Solve the initial value problem*

$$\frac{d^2y}{dt^2} + 16y = \cos 25t, \qquad y(0) = 0, \qquad y'(0) = \frac{1}{100}.$$

Plot the solution and describe its characteristic features.

4.4 Resonance and oscillators

In this section we continue our study of oscillators with trigonometric forcing. Our goal is to understand what happens when the forcing frequency approaches the natural frequency of the oscillator. To do this, we construct an oscillator equation with natural frequency ω and forcing frequency ω_f as follows:

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega_f t). \tag{4.4.1}$$

Our plan is the following: We consider (4.4.1) with initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0 \tag{4.4.2}$$

in the situation that $\omega_f \neq \omega$. Then we take the limit as $\omega_f \to \omega$ and see what happens. (The reason for choosing initial conditions (4.4.2) is that we want to focus attention on the effects of the forcing.)

It is straightforward to see that the homogeneous solution to (4.4.1) is

$$y_h(t) = \alpha\cos(\omega t) + \beta\sin(\omega t).$$

As in the previous section, we now proceed by looking for a particular solution of the form $y_p(t) = a\cos(\omega_f t)$. Plugging this in to (4.4.1), obtain

$$(\omega^2 - \omega_f^2)a\cos(\omega_f t) = \cos(\omega_f t). \tag{4.4.3}$$

Since we are assuming that $\omega_f \neq \omega$, we obtain the particular solution

$$y_p(t) = \frac{1}{\omega^2 - \omega_f^2}\cos(\omega_f t)$$

and thus see that the general solution is

$$y(t) = \alpha\cos(\omega t) + \beta\sin(\omega t) + \frac{1}{\omega^2 - \omega_f^2}\cos(\omega_f t). \tag{4.4.4}$$

We now enforce the initial conditions (4.4.2), computing

$$y'(t) = -\omega\alpha\sin(\omega t) + \omega\beta\cos(\omega t) - \frac{\omega_f}{\omega^2 - \omega_f^2}\sin(\omega_f t).$$

Thus the initial conditions require

$$0 = \alpha + \frac{1}{\omega^2 - \omega_f^2} \qquad \text{and} \qquad 0 = \beta.$$

Consequently, the solution to (4.4.1) – (4.4.2) is

$$y(t) = \frac{\cos(\omega_f t) - \cos(\omega t)}{\omega^2 - \omega_f^2}. \tag{4.4.5}$$

We now want to take the limit of (4.4.5) as $\omega_f \to \omega$. Notice that both the numerator and the denominator are zero in the limit. Thus it is appropriate to apply l'Hôpital's rule. Keeping in mind that the variable with which we are taking the limit is ω_f, we find that

$$\lim_{\omega_f \to \omega} \left[\frac{\cos(\omega_f t) - \cos(\omega t)}{\omega^2 - \omega_f^2} \right] = \lim_{\omega_f \to \omega} \left[\frac{-t \sin(\omega_f t)}{-2\omega_f} \right] = \frac{t}{2\omega} \sin(\omega t).$$

Thus in the limit as the forcing frequency approaches the natural frequency, the solution approaches a sine wave that oscillates at the natural frequency and has a linearly growing amplitude.

In order to understand what's going on here, it is useful to recall the concept of *beats* from Example 4.3.3. In that example, we used a trigonometric identity to rewrite the solution in a way that we could interpret as oscillations at the natural frequency with oscillating amplitude. In order to apply that approach here, we use the identity

$$\cos(A) - \cos(B) = 2 \sin\left(\frac{B+A}{2}\right) \sin\left(\frac{B-A}{2}\right)$$

in order to write the solution (4.4.5) as

$$y(t) = \frac{1}{\omega^2 - \omega_f^2} \sin\left(\frac{\omega_f - \omega}{2} t\right) \sin\left(\frac{\omega_f + \omega}{2} t\right).$$

We interpret this last expression as being sinusoidal oscillations of frequency $(\omega_f + \omega)/2$ with periodic amplitude given by

$$\frac{2}{\omega^2 - \omega_f^2} \sin\left(\frac{\omega_f - \omega}{2} t\right).$$

Thus as the forcing frequency approaches the natural frequency, we see that the frequency of the oscillations approaches the natural frequenc; and that the magnitude of the amplitude function increases, while the frequency of the amplitude function decreases. In other words, as ω_f approaches ω, the solution (4.4.5) consists of increasingly large and slow beats of oscillations at near-natural frequency. In the limit, the first beat "takes over" the solution and we an amplitude function that is simply growing linearly.

The following code demonstrates the limit as $\omega_f \to \omega$ nicely:

```
var('t')
@ interact
def f(wf = slider(1.0000001,1.5, default=1.5, label='$\omega_f$')):
    solnplot = plot( ( cos(wf*t) - cos(t) )/(1-wf^2), (t,0,120))
    ampplot1 = plot( 2*sin( t*(wf-1)/2)/(1-wf^2)  , (t,0,120), linestyle
        ='dashed', color='red',thickness=2)
    ampplot2 = plot( -2*sin( t*(wf-1)/2)/(1-wf^2)  , (t,0,120),
        linestyle='dashed', color='red',thickness=2)
    mainplot = solnplot + ampplot1 + ampplot2
    mainplot.show(ymin=-10,ymax=20)
```

The code can also be accessed via this link.

The picture generated by the code above illustrates the consequences of changing the forcing frequency: As the forcing frequency becomes the closer the natural frequency, the natural response of the system grows dramatically in size, ultimately approaching the linearly growing function

$$y_p(t) = \frac{t}{2\omega}\sin(\omega t). \tag{4.4.6}$$

The situation where the forcing frequency is the same as the natural frequency is an example of ***resonance***. Physically, resonance is when the forcing is "tuned" to the natural frequency of the system. This results in oscillations with amplitudes that grow.

Mathematically, we see resonance occur when there is some sort of degeneracy in the system. In the case of the forced oscillator in which the forcing frequency matches the natural frequency, this degeneracy manifests itself in the fact that we do not find a particular solution of the same form as the forcing. Rather, we find a particular solution that is of the form $t \cdot y_h(t)$. This is explored in greater detail in the next section.

Exercise 4.4.1. *In this problem we consider directly the forced oscillator equation (4.4.1) when $\omega_f = \omega$:*

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega t). \tag{4.4.7}$$

1. *Show that looking for a particular solution to (4.4.7) of the form $y_p(t) = a\cos(\omega t)$ yields an equation that cannot be satisfied.*

2. *Show by direct computation that (4.4.6) is actually a particular solution to (4.4.7).*

3. *Find the general solution to (4.4.7).*

Exercise 4.4.2. *Find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 3\cos(2t).$$

Exercise 4.4.3. *In this exercise, we study another type of resonance Consider the equation*

$$\frac{d^2y}{dt^2} - 2b\frac{dy}{dt} + y = 0, \tag{4.4.8}$$

where b is some parameter with $0 \leq b \leq 1$.

1. *Find the general solution to (4.4.8) when $b = 0$.*
2. *Now assume that $0 < b < 1$. Find the solution to (4.4.8) satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$.*
3. *Show that in the limit as $b \to 1$ we have $y(t) \to te^t$.*
4. *Now set $b = 1$ in the equation (4.4.8) and verify that te^t is a particular solution.*
5. *Find the general solution to (4.4.8) in the case when $b = 1$.*
6. *In the case that $b = 1$, we say that (4.4.8) is resonant. Why is the term "resonant" appropriate in this case?*

Exercise 4.4.4. *Find the general solution to*

$$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = 0.$$

4.5 General resonance

As alluded to in the previous section, the mathematical phenomenon of resonance corresponds to a degeneracy in the system. In this section we explore two different types of degeneracies: repeated eigenvalues and degenerate forcing. In both cases, the degeneracy appears when the "usual" methods yield fewer solutions than we were expecting to find. In both cases, we can find the "remaining" solutions in the form of

$$t \cdot (\text{those solutions we were able to find}).$$

I refer to this method as the "t-trick."

The first type of resonance we study is the case of repeated eigenvalues. An example of this was explored in Exercise 4.4.3. More generally, suppose we have a differential equation of the form

$$\frac{d^2y}{dt^2} - 2\mu\frac{dy}{dt} + \mu^2 y = 0. \tag{4.5.1}$$

The characteristic equation for this ODE is

$$\lambda^2 - 2\mu\lambda + \mu^2 = 0,$$

which we write as

$$(\lambda - \mu)^2 = 0.$$

Thus there is only one eigenvalue, namely $\lambda = \mu$. From this we know that $y_1(t) = e^{\mu t}$ is one solution to (4.5.1). However, in order to use the superposition principle to find the general solution we need to find a second solution. Motivated by Exercise 4.4.3, we investigate whether $te^{\mu t}$ is a solution. We compute

$$\frac{d^2}{dt^2}\left[te^{\mu t}\right] - 2\mu\frac{d}{dt}\left[te^{\mu t}\right] + \mu^2\left(te^{\mu t}\right) = 0$$

and thus conclude that $y_2(t) = te^{\mu t}$ is indeed a solution, and therefore that the general solution to (4.5.1) is

$$y(t) = \alpha e^{\mu t} + \beta te^{\mu t}.$$

Example 4.5.1. *Suppose we want to solve the initial value problem*

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 3.$$

If we look for solutions of the form $e^{\lambda t}$ we find that λ must satisfy the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0.$$

The only such λ is $\lambda = 4$ and thus we find only the solution e^{4t}. However, the function te^{4t} is also a solution, and thus by superposition the general solution is

$$y(t) = \alpha e^{4t} + \beta t e^{4t}.$$

In preparation for imposing the initial conditions, we compute

$$y'(t) = 4\alpha e^{4t} + \beta e^{4t} + 4\beta t e^{4t}.$$

Thus the initial conditions reduce to

$$1 = \alpha \qquad \text{and} \qquad 3 = 4\alpha + \beta.$$

Consequently, we find that the solution to the initial value problem is

$$y(t) = e^{4t} - te^{4t}.$$

Activity 4.5.1. *Find the solution to the initial value problem*

$$9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + y = 7, \qquad y(0) = 1, \qquad y'(0) = 5.$$

The second type of resonance we consider is forced equations for which our "usual guess" for the particular solution does not work. This happens when, for example, the forcing term is of the same term as the homogeneous solution. When this occurs, we make the "educated guess" that the particular solution takes the form $t \cdot y_h(t)$.

Example 4.5.2. *Consider the differential equation*

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = e^{-2t}.$$

We first address the homogeneous equation

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

by solving the characteristic equation

$$\lambda^2 + 7\lambda + 10 = 0.$$

We find that $\lambda = -2$ and $\lambda = -5$ are solutions, from which we deduce that the homogeneous solution is

$$y_h(t) = \alpha e^{-2t} + \beta e^{-5t}.$$

Notice that the forcing function $f(t) = e^{-2t}$ is of the same form as the homogeneous solution. In particular, if we go looking for a particular solution of the form $y_p(t) = ae^{-2t}$ we obtain the equation

$$0 = e^{-2t},$$

which is a contradiction.

Instead, we use the t-trick, and go looking for a particular solution of the form

$$y_p(t) = ate^{-2t}.$$

Plugging this in to the original equation yields

$$\frac{d^2}{dt^2}\left[ate^{-2t}\right] + 7\frac{d}{dt}\left[ate^{-2t}\right] + 10\left(ate^{-2t}\right) = e^{-2t},$$

which simplifies to

$$3ae^{-2t} = e^{-2t}.$$

Thus we choose $a = 1/3$ and have particular solution

$$y_p(t) = \frac{1}{3}te^{-2t}.$$

Using this, we find the general solution

$$y(t) = \alpha e^{-2t} + \beta e^{-5t} + \frac{1}{3}te^{-2t}.$$

Sometimes we get to use the t-trick twice, as the following activity illustrates!

Activity 4.5.2. *Find the general solution to the equation*

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^{2t}.$$

Exercises exercises

Exercise 4.5.1. *Find the general solution of the following ODEs:*

1. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0;$

2. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 1 + t + e^{2t};$

3. $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0;$

4. $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = e^{4t};$

5. $\frac{d^2y}{dt^2} + 9y = \sin(2t);$

6. $\frac{d^2y}{dt^2} + 9y = 3\cos(3t).$