Chapter 6

Synthesis and applications

6.1 Practice using the Fundamental Theorems of Calculus

Exercises.

1. Please compute the following.

   (a) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V} = \langle y, 3x \rangle$ and where C is the counter-clockwise unit circle centered at the origin.

   (b) The inward flux of $\vec{V}(x, y) = \langle x, y - y^2 \rangle$ across the unit circle centered at $(1, 0)$.

   (c) The outward flux of $\vec{V} = \langle 3x, y \rangle$ across the top half of the unit circle centered at $(1, 0)$.

   (d) The outward flux of $\vec{V} = \langle x, -y, z \rangle$ across the surface of the sphere of radius 2 centered at the origin.

   (e) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V} = \langle x, y - z, y + z \rangle$ and where C is the North 60-th parallel of the unit sphere centered at the origin, oriented counterclockwise when viewed from the above.

6.2 Changing the domain of integration

Key Ideas.

- *Gauss' Theorem allows us to compare the flux of* $\vec{V}$ *across different curves/surfaces. If* $\operatorname{div}(\vec{V}) = 0$ *in the region between, then the flux is the same.*
- *Green's Theorem and Stokes' Theorem allow us to compare the work done by* $\vec{V}$ *along different paths. If* $\operatorname{curl}(\vec{V}) = 0$ *in the surface connecting the paths, then the work is the same.*
- *It is important to be aware of "point singularities." Classic examples are:*

$$\frac{1}{r^2}\partial_\theta = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle, \qquad \frac{1}{r}\partial_r = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

$$\frac{1}{r^2}\partial_r = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

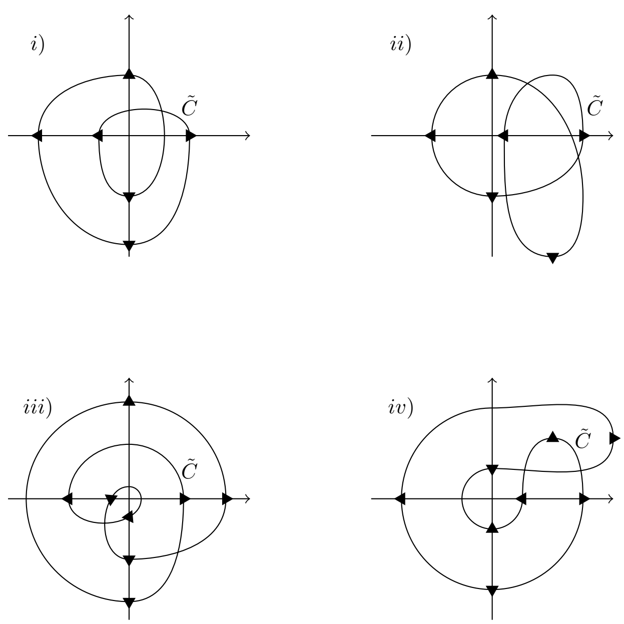
Exercises.

1. In this problem you study the work done by the vector field $\frac{1}{r}\partial_r$ along a simple counter-clockwise closed curve (contour) C.

   (a) What is the value of $\int_C \vec{V} \cdot \hat{N}\, ds$ if C does not contain the origin in its interior? Provide a computation / justification for your claim.

   (b) What is the value of $\int_C \vec{V} \cdot \hat{N}\, ds$ if C does contain the origin in its interior? Provide a computation / justification for your claim.

   (c) Furthermore, what is the value of $\int_{\tilde{C}} \vec{V} \cdot \hat{N}\, ds$ for the following contours $\tilde{C}$:

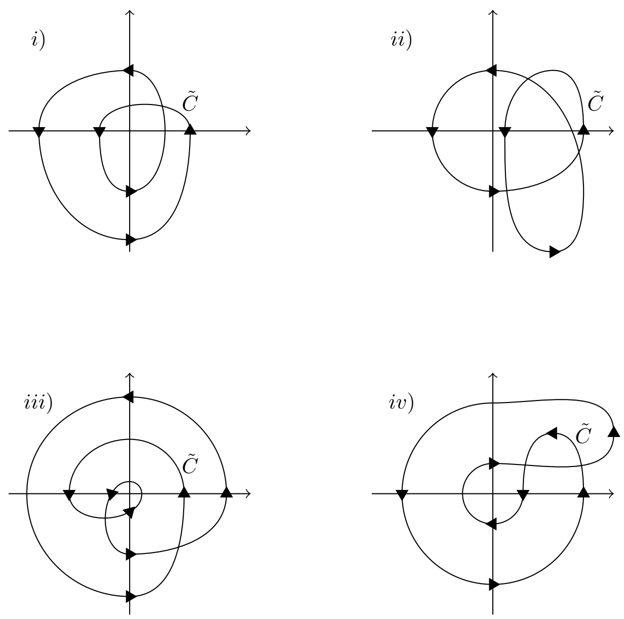


2. In this problem you study the flux of the vector field $\frac{1}{r^2}\partial_\theta$ along a simple counter-clockwise closed curve (contour) C with an outward-pointing normal $\hat{N}$.

    (a) What is the value of $\int_C \vec{V}\cdot\hat{T}\,ds$ if C does not contain the origin in its interior? Provide a computation / justification for your claim.

    (b) What is the value of $\int_C \vec{V}\cdot\hat{T}\,ds$ if C does contain the origin in its interior? Provide a computation / justification for your claim.

    (c) Furthermore, what is the value of $\int_{\tilde{C}} \vec{V}\cdot\hat{T}\,ds$ for the following contours $\tilde{C}$:

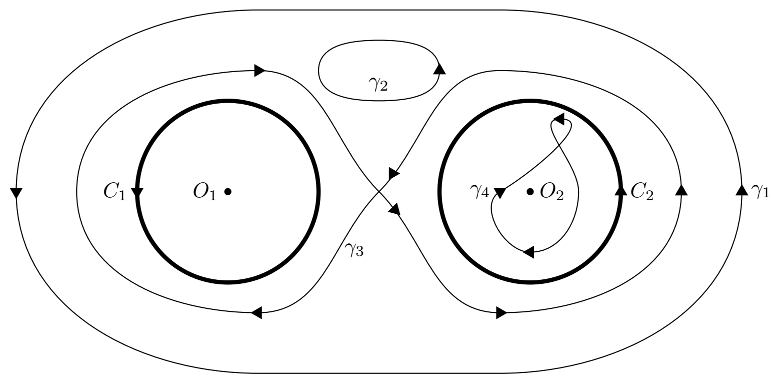


3. Let C_1 be the ellipse $4x^2+9y^2=1$ and let C_2 be the standard unit circle. Assume both are oriented counterclockwise. Furthermore, let

$$\vec{V}(x,y)=\left\langle \frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2}\right\rangle.$$

Use Green's theorem to argue that $\int_{C_1}\vec{V}\cdot\hat{T}\,ds=\int_{C_2}\vec{V}\cdot\hat{T}\,ds$. Based on this find the value of $\int_{C_1}\vec{V}\cdot\hat{T}\,ds$.

4. Suppose that a vector field $\vec{V}(x,y)=\langle P(x,y),Q(x,y)\rangle$ is defined and smooth everywhere except at points O_1 and O_2, and that $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ (i.e. that $\vec{V}$ is "curl-free"). Let C_1 and C_2 be circles centered at points O_1 and O_2 respectively, and let γ_1, γ_2, γ_3 and γ_4 be the closed curves sketched below. (Orientation is also indicated on the sketch.) (Also, γ_3 looks like a figure-eight.)



Finally, suppose that

$$\int_{C_1} \vec{V}\cdot\hat{T}\,ds = 1 \quad\text{and}\quad \int_{C_2} \vec{V}\cdot\hat{T}\,ds = 3.$$

Compute the circulations $\int_{\gamma_1} \vec{V}\cdot\hat{T}\,ds$, $\int_{\gamma_2} \vec{V}\cdot\hat{T}\,ds$, $\int_{\gamma_3} \vec{V}\cdot\hat{T}\,ds$ and $\int_{\gamma_4} \vec{V}\cdot\hat{T}\,ds$.

5. (a) Let C denote the upper half of the ellipse $4x^2 + 9y^2 = 16$. Consider the vector field $\vec{V}(x,y) = \langle 1+2x+3y, -x-2y\rangle$; it is easy to see that $\operatorname{div}(\vec{V}) = 0$. Compute the outward flux integral $\int_C \vec{V}\cdot\hat{N}\,ds$. (Note: the "diameter" at the bottom is not included in C.)

   (b) Let C denote the circle of radius 2 centered at the origin. Consider the vector field

$$\vec{V}(x,y) = \left\langle \frac{1-x}{(x-1)^2+(y-1)^2}, \frac{1-y}{(x-1)^2+(y-1)^2} \right\rangle;$$

   a computation shows that $\operatorname{div}(\vec{V}) = 0$. Find the outward flux integral $\int_C \vec{V}\cdot\hat{N}\,ds$.

6. Consider the vector field given in spherical coordinates by

$$\vec{V} = \frac{1}{r^2}\partial_r;$$

in an earlier homework assignment you computed that $\operatorname{div}(\vec{V}) = 0$.

   (a) Given the fact that $\vec{V}$ is undefined at the origin, discuss the appropriate ways of applying the Gauss' Theorem to (the flux of) the vector field $\vec{V}$.

   (b) What is the outward flux of the vector field $\vec{V}$ across the unit sphere centered at the origin?

   (c) What is the outward flux of the vector field $\vec{V}$ across some other sphere centered at the origin?

   (d) What is the outward flux of the vector field $\vec{V}$ across some other surface which contains the origin in its interior?

   (e) What is the outward flux of the vector field $\vec{V}$ across some other surface which contains the origin in its exterior?

7. Let S be the surface of the cylinder of radius 1 centered along the z-axis, bounded between the xy-plane on the bottom and the plane $x + y + z = 20$ on the top. Note that the bases (bottom, lid) are not included in S. Let C_1 be the bottom boundary of the cylinder S (i.e. the unit circle in the xy-plane), and let C_2 be the top boundary of the cylinder S (i.e. the ellipse in the slanted pane $x+y+z = 20$). Let both C_1 and C_2, when viewed from above, be oriented clockwise. Finally, let $\vec{V}(x, y, z) = \langle x - y, x + y, z\rangle$.

   (a) Make a rough sketch of S, C_1 and C_2. Please include orientation arrows.

   (b) Apply Stokes' Theorem to the situation. You should get an equality involving flux and circulation integrals.

   (c) Compute the flux integral appearing in the equality you learned about in part (b).

   (d) Compute the easier one of the two circulation integrals appearing in the equality you learned about in part (b).

   (e) What is the value of $\int_{C_2} \vec{V} \cdot \hat{T}\, ds$?

6.3 More practice using the Fundamental Theorems

Exercises.

1. Please compute $\int_C \vec{V} \cdot \hat{T}\, ds$ where

   (a) $\vec{V}(x, y) = \langle -y^3, x^3\rangle$ and where C is the counter-clockwise unit circle centered at the origin.

   (b) $\vec{V}(x, y) = \langle y, x\rangle$ and where C is the counter-clockwise ellipse $4x^2 + y^2 = 1$.

   (c) $\vec{V}(x, y) = \langle \frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2}\rangle$ and where C is the ellipse $4x^2 + 9y^2 = 1$ oriented counter-clockwise.

   (d) $\vec{V}(x, y, z) = \langle -z, -z, x + y\rangle$ and where C is the North 45-th parallel of the unit sphere centered at the origin, oriented counterclockwise when viewed from the above.

   (e) $\vec{V}(x, y, z) = \langle z + x, x + y, y + z\rangle$ and where C is the contour of the triangle with edges $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in that order.

   (f) $\vec{V}(x, y, z) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\rangle$ and where C is a slanted ellipse centered somewhere on the z-axis.

2. Please compute the following flux integrals.

   (a) The outward flux of $\vec{V}(x, y) = \langle 1, 1\rangle$ across the boundary of some closed contour C;

   (b) The inward flux of $\vec{V}(x, y) = \langle 2xy,\, x^2 + y^2\rangle$ across the boundary of the rectangle with vertices at $(0, 0)$, $(1, 0)$, $(1, 2)$, $(0, 2)$ oriented counter-clockwise.

   (c) The outward flux of the vector field $\vec{V}(x, y) = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\rangle$ across the ellipse $x^2+4y^2 = 1$;

   (d) The outward flux of the vector field $\vec{V}(x, y) = \langle x^3, y^3\rangle$ across the top half of the unit circle centered at the origin;

   (e) The outward flux of $\vec{V}(x, y, z) = \langle x^3+y^3, y^3+z^3, z^3+x^3\rangle$ across the unit sphere centered at the origin;

   (f) The inward flux of $\vec{V}(x, y, z) = \langle xy, yz, xz\rangle$ across the bowl of the paraboloid $z = x^2+y^2$ up to and including the lid at $z = 4$;

   (g) The same as the previous problem except with the lid removed;

   (h) The inward flux of $\vec{V}(x, y, z) = \langle x^2 + x, y^2 + y, -z^2 - z\rangle$ across the surface of the cone $z = \sqrt{x^2 + y^2}$ up to but not including the lid at $z = 2$;

   (i) The outward flux of $\vec{V}(x, y, z) = \langle x, y, z\rangle$ across the cube $-1 \le x, y, z \le 1$.

   (j) The outward flux of $\vec{V}(x, y, z) = \langle 1, 1, 1\rangle$ across the upper hemisphere of the unit sphere centered at the origin.

6.4 The Fundamental Theorem of Calculus in Gradient Form

Key Ideas.

- $\vec{V}$ *a* ***gradient vector field*** *if* $\vec{V} = \text{grad}(u)$ *for some function* u*. In this case, the function* u *is called a* ***potential*** *for* $\vec{V}$*.*
- *If* $\vec{V}$ *has a potential function, then we can use the Fundamental Theorem for gradients to compute* $\int_C \vec{V} \cdot \hat{T}\, ds$*.*
- *Important examples include*

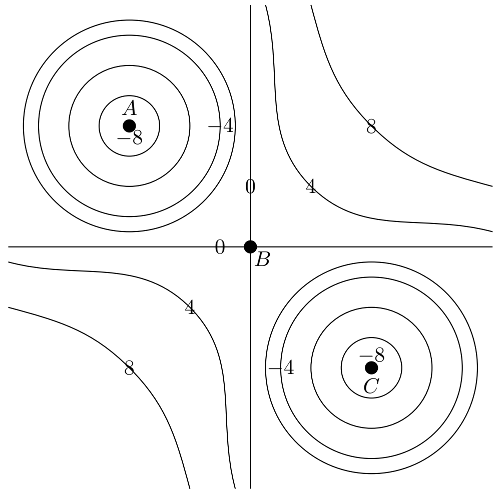
$$u(x,y) = \frac{1}{2}\ln(x^2+y^2) \quad \leftrightarrow \quad \text{grad}(u) = \frac{1}{r}\partial_r$$
$$u(x,y,z) = (x^2+y^2+z^2)^{-1/2} \quad \leftrightarrow \quad \text{grad}(u) = \frac{1}{r^2}\partial_r$$

- *If* $\text{curl}(\vec{V}) \neq 0$ *then there cannot exist a potential function for* $\vec{V}$*. The reason is that* $\text{curl}(\text{grad}(u)) = 0$ *for all functions* u*.*

Exercises.

1. Are the following vector fields $\vec{V}$ gradient vector fields? If so, find their potential.
   (a) $\vec{V}(x,y) = \langle x+y, x+y\rangle$;
   (b) $\vec{V}(x,y) = \langle x+2y, x+2y\rangle$;
   (c) $\vec{V}(x,y,z) = \langle x+y+z, x+y+z, x+y+z\rangle$.
2. Compute the following integrals. Use the Fundamental Theorem of Vector Calculus in gradient form whenever appropriate.
   (a) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V}(x,y) = \langle x, y\rangle$ and where C is the line segment going from $(1,0)$ to $(1,1)$.
   (b) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V}(x,y,z) = \langle x+y+z, x+y+z, x+y+z\rangle$ and where C is a counter-clockwise helix going from $(1,0,0)$ to $(1,0,4\pi)$.
   (c) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V} = \langle \cos(y) - z\sin(x), \cos(z) - x\sin(y), \cos(x) - y\sin(z)\rangle$ and where C is the counter-clockwise helix going from $(1,0,0)$ to $(1,0,4\pi)$.
   (d) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V}(x,y,z) = \langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\rangle$ and where C is a counter-clockwise helix going from $(1,0,0)$ to $(1,0,4\pi)$.
   (e) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V}(x,y,z) = \langle x, x+y, x+y+z\rangle$ where C is the line segment joining $(1,1,0)$ and $(0,0,1)$.
   (f) $\int_C \vec{V} \cdot \hat{T}\, ds$ where $\vec{V}(x,y) = \langle y, x\rangle$ and where C is the top half of the counter-clockwise ellipse $4x^2 + y^2 = 1$.

   The following is the contour map of $f(x,y)$. Please note the contour labels, particularly at the points A, B and C.



(a) Sketch grad(f).

(b) Evaluate $\int_\gamma \text{grad}(f) \cdot \hat{T}\, ds$ where the curve γ

i. goes from A to B;

ii. goes from A to C.

(c) Estimate the $\pm$ sign of div(grad(f)) at points A and C.

3. Consider the vector field

$$\vec{V}(x, y) = \langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle$$

(a) Verify that both $f_1(x, y) = \arctan\left(\frac{y}{x}\right)$ and $f_2(x, y) = -\arctan\left(\frac{x}{y}\right)$ serve as potentials for the vector field $\vec{V}$.

(b) What are the domains for f_1 and f_2?

(c) Let P_1, P_2, P_3 and P_4 denote the points $(1, -1)$, $(1, 1)$, $(-1, 1)$ and $(-1, -1)$, respectively. Furthermore, let P_1P_2 denote the line segment from P_1 to P_2, let P_2P_3 denote the line segment from P_2 to P_3, etc. Which one of the following is true, and why?

i. $\int_{P_1P_2} \vec{V} \cdot \hat{T}\, ds = f_1(P_2) - f_1(P_1)$ or $\int_{P_1P_2} \vec{V} \cdot \hat{T}\, ds = f_2(P_2) - f_2(P_1)$?

ii. $\int_{P_2P_3} \vec{V} \cdot \hat{T}\, ds = f_1(P_3) - f_1(P_2)$ or $\int_{P_2P_3} \vec{V} \cdot \hat{T}\, ds = f_2(P_3) - f_2(P_2)$?

iii. $\int_{P_3P_4} \vec{V} \cdot \hat{T}\, ds = f_1(P_4) - f_1(P_3)$ or $\int_{P_3P_4} \vec{V} \cdot \hat{T}\, ds = f_2(P_4) - f_2(P_3)$?

iv. $\int_{P_4P_1} \vec{V} \cdot \hat{T}\, ds = f_1(P_1) - f_1(P_4)$ or $\int_{P_4P_1} \vec{V} \cdot \hat{T}\, ds = f_2(P_1) - f_2(P_4)$?

(d) Based on part (c) alone, find the value of the circulation integral $\int_{P_1P_2P_3P_4} \vec{V} \cdot \hat{T}\, ds$ along the (counterclockwise) rectangular path $P_1P_2P_3P_4$ from above?