Writing Assignment 1: Comparing Harvesting Models

This is Paul's version of Writing Assignment 1.

The logistic model with harvesting

First, recall the the logistic population model, which assumes that the population lives in a habitat that can sustainably support a population of size K. We further assume that the relative growth rate is proportional to the percent of available habitat. These two assumptions lead to the differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

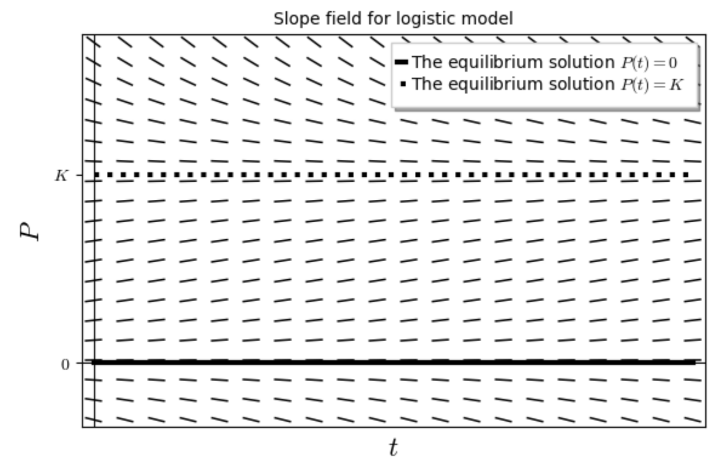
where r corresponds to the relative growth rate of the population in the absense of habitat constraints.

The logistic model has two equilibrium solutions: $P(t) = 0$ is unstable, while $P(t) = K$ is stable. The slope field plot below shows the two equilibrium solutions.

In [29]:
```
var('y','t')

sfplot=plot_slope_field(y*(1-y),(t,0,2), (y,-.3,1.7),
    title="Slope field for logistic model")
eq0plot = plot(0, (t,0,2),thickness=3, color="black", linestyle="-",
    legend_label="The equilibrium solution $P(t) =0$")
eq1plot = plot(1, (t,0,2),thickness=3, color="black", linestyle=':',
    legend_label="The equilibrium solution $P(t) =K$")
```

In [30]:
```
mainplot = sfplot + eq0plot + eq1plot
mainplot.show(axes_labels = ["$t$","$P$"],
    ticks=[[],[0,1]],
    tick_formatter = [None,["$0$", "$K$"]])
```



We consider two versions of the logistic model with harvesting. In the first model, we suppose that the there is constant harvesting (or removal from the population) at a constant rate c. This assumption leads to the differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - c.$$

We refer to this model as the *constant harvesting model*.

In the second model, we assume that harvesting is done at a constant relative rate. That is, there is a fixed percent of the population that is removed per unit time. This assumption leads to the differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - bP,$$

where b is the precent removed per unit time. We refer to this model as the *percent harvesting model*.

The constant harvesting model

The constant harvesting model has two equilibrium solutions $P(t) = P_+$ and $P(t) = P_-$, where the constants $P_\pm$ are given by

$$P_\pm = \frac{K \pm K\sqrt{1 - \frac{4c}{Kr}}}{2}.$$

From this formula we see that the value of the constant $\frac{4c}{Kr}$ is important.

- If $\frac{4c}{Kr} > 1$ then in fact no equilibrium solutions exist at all.
- If $\frac{4c}{Kr} = 1$ then there is a single equilibrium solution $P(t) = K/2$.
- If $\frac{4c}{Kr} < 1$ then there are two positive equilibrium solutions $P(t) = P_+$ and $P(t) = P_-$.

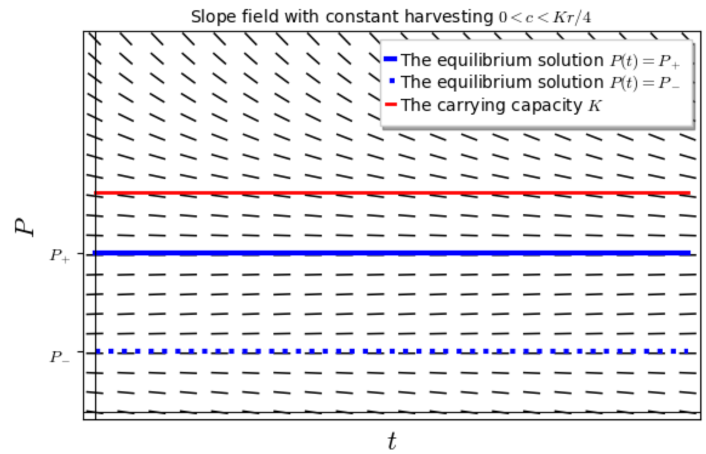
This tells that in order to have an equilibrium solution, the harvesting rate c must satisfy $c \leq Kr/4$.

We now address the stability of the equilibrium solutions. The following plot shows the case when $0 < c < \frac{Kr}{4}$ and thus there are two equilibrium solutions. From the plot we see that the equilibrium $P(t) = P_+$ is stable, while the equilibrium $P(t) = P_-$ is unstable. Furthermore, we see that the equilibrium $P(t) = P_+$ is less than the carrying capacity.

In [31]:
```
c = .2
eq1 = (1+sqrt(1-4*c))/2
eq2 = (1-sqrt(1-4*c))/2

sfplot=plot_slope_field(y*(1-y)-c,(t,0,2), (y,0,1.7),
    title="Slope field with constant harvesting $0< c < Kr/4$")
plotK = plot(1, (t,0,2), thickness=2, color="red", linestyle="-",
    legend_label="The carrying capacity $K$")
eq1plot = plot(eq1, (t,0,2), thickness=3, color="blue", linestyle="-",
    legend_label="The equilibrium solution $P(t) =P_+$")
eq2plot = plot(eq2, (t,0,2), thickness=3, color="blue", linestyle=':',
    legend_label="The equilibrium solution $P(t) =P_-$")
```

In [32]:
```
mainplot = sfplot+eq1plot+eq2plot+plotK
mainplot.show(axes_labels = ["$t$","$P$"],
    ticks=[[],[eq2,eq1]],
    tick_formatter = [None,["$P_-$", "$P_+$"]])
```

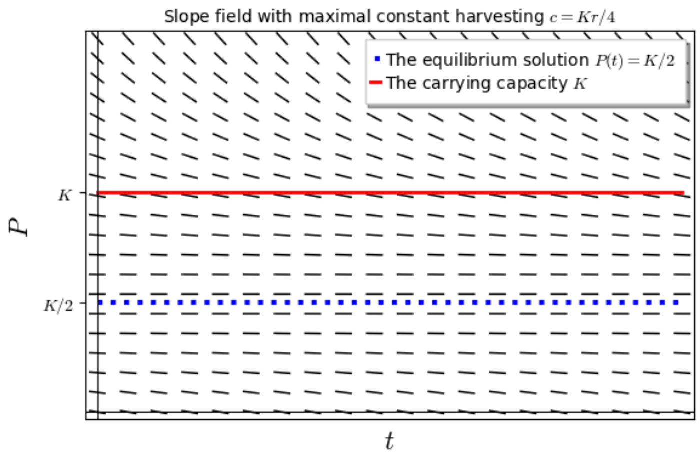


In case the extreme case that $c = \frac{Kr}{4}$ there is only one equilibrium solution: $P(t) = \frac{K}{2}$. The following plot shows that in this case the equilibrium is unstable.

In [33]:
```
c = .25
eq = 1/2

sfplot=plot_slope_field(y*(1-y)-c,(t,0,2), (y,0,1.7),
    title="Slope field with maximal constant harvesting $c = Kr/4$")
plotK = plot(1, (t,0,2), thickness=2, color="red", linestyle="-",
    legend_label="The carrying capacity $K$")
eqplot = plot(eq, (t,0,2), thickness=3, color="blue", linestyle=":",
    legend_label="The equilibrium solution $P(t) =K/2$")
```

In [34]:
```
mainplot = sfplot+eqplot+plotK
mainplot.show(axes_labels = ["$t$","$P$"],
    ticks=[[],[eq,1]], tick_formatter = [None,["$K/2$","$K$"]], legend_l
oc=1)
```



We summarize the situation for the constant harvesting model as follows. There is a maximum harvesting rate $\frac{Kr}{4}$. If the harvesting rate exceeds this maximum rate, then there is no equilibrium solution to the model. If the harvesting rate is equal to the maximum, then there is one unstable equilibrium solution. Finally, if the harvesting rate is less than the maximum, there there are two equilibrium solutions, one of which is stable. Thus under this harvesting strategy, it is not advisable to harvest the theoretical maximum amount.

The percent harvesting model

The percent harvesting model has two equilibrium solutions: $P(t) = 0$ and $P(t) = P_*$, where

$$P_* = K\left(1 - \frac{b}{r}\right).$$

In order for the equilibrium solution $P(t) = P_*$ to be positive we must have $b < r$. If $b \geq r$ then there exists no positive equilibrium solution.

Using this expression for P_* we can algebraically re-arrange the percent harvesting model in to the following form

$$\frac{dP}{dt} = \frac{r}{K}P\,(P_* - P)\,.$$

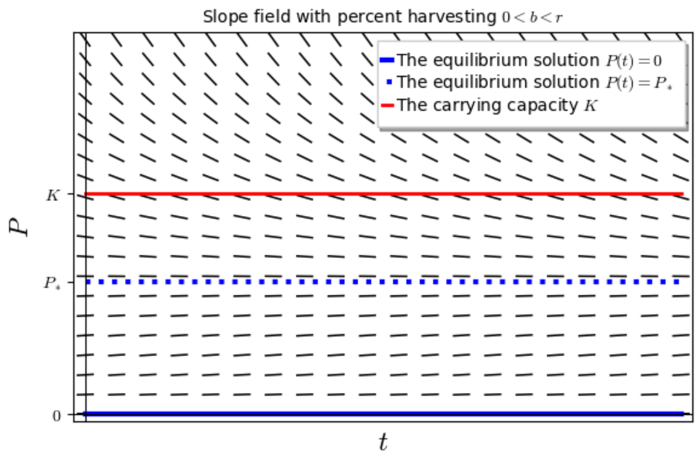
From this version of the model, it is easy to deduce that the equilibrium solution $P(t) = P_*$ is stable so long as $b < r$ (which ensures that $P_* > 0$).

The following plot shows the slopefield for this model.

In [35]:
```
b = .4
eq1 = 0
eq2 = 1-b

slopefield0=plot_slope_field(y*(1-y)-b*y,(t,0,2), (y,0,1.7),
    title="Slope field with percent harvesting $0< b < r$")
plotK = plot(1, (t,0,2), thickness=2, color="red", linestyle="-",
    legend_label="The carrying capacity $K$")
equilibrium1 = plot(eq1, (t,0,2), thickness=3, color="blue", linestyle=
"-",
    legend_label="The equilibrium solution $P(t) =0$")
equilibrium2 = plot(eq2, (t,0,2), thickness=3, color="blue",  linestyle=
':',
    legend_label="The equilibrium solution $P(t) =P_*$")
```

```
In [36]: mainplot = slopefield0+equilibrium1+equilibrium2+plotK
mainplot.show(axes_labels = ["$t$","$P$"],
    ticks=[[],[eq1,eq2,1]], tick_formatter = [None,["$0$", "$P_*$","$K$"
]])
```



We now turn to the question of maximizing the harvest for the percent harvesting model. We want our maximum harvest to be at equilibrium, so our goal is to find the value of b that makes the quantity bP_* the largest. Note that

$$bP_* = \frac{K}{r}b(r-b).$$

If we view this as a function of b and look for the maximum we find that the value of b that makes bP_* the largest is $b_* = r/2$. Therefore the maximum harvest at equilibrium is

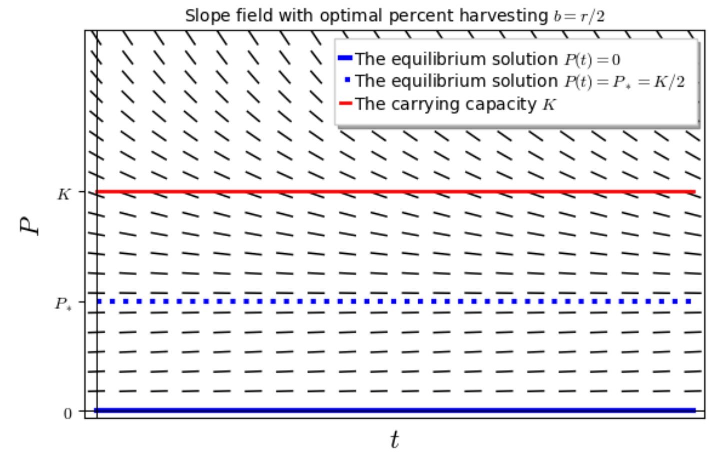
$$b_*P_* = \frac{Kr}{4}.$$

With this choice of b, the slope field plot is as follows.

```
In [37]: b = .5
eq1 = 0
eq2 = 1-b

sfplot=plot_slope_field(y*(1-y)-b*y,(t,0,2), (y,0,1.7),
    title="Slope field with optimal percent harvesting $b = r/2$")
plotK = plot(1, (t,0,2), thickness=2, color="red", linestyle="-",
    legend_label="The carrying capacity $K$")
eq1plot = plot(eq1, (t,0,2), thickness=3, color="blue", linestyle="-",
    legend_label="The equilibrium solution $P(t) =0$")
eq2plot = plot(eq2, (t,0,2), thickness=3, color="blue",  linestyle=':',
    legend_label="The equilibrium solution $P(t) =P_* = K/2$")
```

```
In [38]: mainplot = sfplot+eq1plot+eq2plot+plotK
mainplot.show(axes_labels = ["$t$","$P$"],
    ticks=[[],[eq1,eq2,1]],
    tick_formatter = [None,["$0$", "$P_*$","$K$"]])
```



We summarize our analysis of the percent harvesting model as follows. Provided the relative harvesting rate is less than the ideal growth rate, the model has a stable equilibrium. In order to have the largest possible harvesting rate, one must choose the relative harvesting rate to be precisely half of the ideal growth rate. With this choice of harvesting rate, the equilibrium population is half of the carrying capacity.

Discussion

The analysis above shows that for the constant harvesting model, the largest harvesting rate for which the model has an equilibrium solution is $c = Kr/4$. In this case, the equilibrium population is $K/2$. This equilibrium solution, however, is not stable. (In fact, it is semistable.)

For the precent harvesting model, there is a positive equilibrium solution provided the relative harvesting rate b satisfies $b < r$. A choice of $b = r/2$ leads to the largest possible harvesting rate. With this choice of b, the equilibrium solution is $P(t) = K/2$. This equilibrium solution is stable.

Both models permit a maximal choice of harvesting. In both cases, the equilibrium population is $K/2$ and in both cases the maximum harvesting rate at equilibrium is $Kr/4$. For the percent harvesting model this equilibrium is stable, while for the constant harvesting model this equilibrium is not stable.

Thus one can achieve the same optimal harvest with either model, but the stability of the situation depends on how one goes about obtaining that optimal harvest.