Writing assignment: Gravitation

Goals for this assignment

- Math: Use conservation of energy to analyze a physically interesting system.
- Technology: Use Sage to generate relevant plots and, if desired, numerical simulations.
- Writing: Craft a coherent narrative that connects background information, mathematical analysis, and relevant graphics.

The "target audience" for this assignment is a student who has taken differential equations, but not completed this assignment. This means that you can assume that the student knows about potential functions and conservation of energy, but is not necessarily familiar with these particular equations.

There is also opportunity to explore a "challenge topic" that goes beyond the basic analysis of the equations below. Three possible challenge topics are listed; you're welcome to choose another challenge topic if you wish.

Grading scheme

Your submissions will be graded on the following:

- Math: Did you use potential functions and conservation of energy to analyze this system? Is your math correct? Do you address the mathematical questions posed?
- Technology: Did you use Sage to generate plots to create illustrations that show off the relevant features of the system? Are these plots correct and well-crafter?
- Writing: Do you correctly present the material in a clear, succinct manner? Do you use good notation, punctuation, grammar, etc.
- Challenge: Were you able to go beyond the basic features of the report in some meaningful way?

If you address the first three parts (math, technology, writing) successfully, then your grade will be a B. If you also address the challenge successfully, then your grade will be an A.

Background information

In this report you explore some simple equations describing the motion of a small object (a planet or comet or astroid) of mass m as it moves relative to a large star of mass M. We assume the planet moves in the equatorial plane of the planet, so that we can describe the motion of the planet by the functions

$r(t)$, describing the distance from the star to the object, and

$\theta(t)$, describing the angular motion of the object.

Note that once you understand the behavior of $r(t)$ and $\theta(t)$ you can understand the motion in Cartesian coordinates using

$$x(t) = r(t)\cos(\theta(t)) \qquad y(t) = r(t)\sin((\theta(t)).$$

According to our friends in the Physics Department, the relevant equations are:

$$m\frac{d^2r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 - \frac{GMm}{r^2}, \tag{3.7.3}$$

$$\frac{d}{dt}\left[mr^2\frac{d\theta}{dt}\right] = 0. \tag{3.7.4}$$

Here G is Newton's gravitational constant.

Your task is to investigate the behavior of the planet predicted by these equations. Before you begin, let me make some remarks:

1. The equation (3.7.4) implies that the quantity $L = mr^2\frac{d\theta}{dt}$ is conserved; L is called the angular momentum. The fact that it is conserved, means that we can treat L as a parameter (just as we treat the energy as a parameter when studying SHO).
2. Knowing that $L = mr^2\frac{d\theta}{dt}$ is conserved means that we can replace $\frac{d\theta}{dt}$ by L/mr^2 in (3.7.3). The result is an equation which only involves r, though it does have four constants: G, M, m, L. For our purposes, we view G and m as fixed, but allow for different values of M and L. Your plan should be to focus on this "reduced" equation for r because once r is determined, θ can be determined by integration:
$$\theta(t) = \theta(t_0) + \int_{t_0}^{t}\frac{L/m}{r(\tau)^2}\,d\tau.$$
3. Since r represents the distance from the star to the planet, we have $r \geq 0$.

Report

Your report is supposed to be an all-inclusive analysis of the behavior of solutions to the system, based on analyzing the reduced equation for r. Your report should include the following elements:

- A short discussion of how to start with the system (3.7.3)-(3.7.4) and obtain a reduced equation for r. The equation will have parameters L, M, G, m in it. Explain how to physically interpret the limits:
$$L/m \to 0 \qquad M \to 0.$$
Explain also what it would mean for $r \to 0$ and for $r \to \infty$.

- Show that the reduced equation for r is actually a Hamiltonian system with some potential function $V(r)$.
- Use the potential function to analyze completely the reduced equation, describing possible scenarios, etc.
- What happens to the potential function in the limit as $L \to 0$ or as $M \to 0$? What is the resulting behavior of r in these special cases?

Challenge topic: Numerical simulation

Based on your potential function, you should be able to construct initial conditions corresponding to different behaviors. Use these initial conditions to construct numerical solutions to the system. Have the computer draw some pretty pictures for you... and comment on what they mean.

Challenge topic: General relativistic motion

In the case that the star of mass M is perfectly spherically symmetric, one can deduce a formula for the energy that takes general relativistic effects into account. The result is that the following energy is conserved:

$$H = \frac{1}{2}v^2 - \frac{M}{r} + \frac{(L/m)^2}{2r^2} - \frac{M(L/m)^2}{r^3}.$$

Here I have followed the convention in general relativity to use units where Newton's constant G and the speed of light c are both dimensionless and equal to 1.

How does the additional "relativistic term" affect the dynamics? For what ranges of L/m and M do the relativistic equations have behavior similar to the Newtonian equations? Etc.

Challenge topic: Kepler's laws

After analyzing a large amount of data, Kepler formulated what we now call "Kepler's laws of planetary motion." The modern formulation of these ideas is:

1. The trajectories of planets form ellipses with the sun at one focus.
2. If one draws a segment connecting the planet to a sun, the area swept out by that segment is constant in time.
3. The orbital period is related (in a specific way) to the size of the ellipse the planet traces out.

In this challenge you connect these three ideas to the equations above. I've given you the rough ideas – your task is to fill in the gaps/details.

1. The first of Kepler's laws requires a significant amount of gymnastics.

   - First we make a change of variables in which we view r as a function of θ rather than a function of t. This is legitimate because (at least for the first time around the orbit) there is a correspondence between time t and angle θ. Using the chain rule and the formula for L we have
   $$\frac{dr}{dt} = \frac{d}{dt} r(\theta(t)) = r'(\theta)\frac{d\theta}{dt} = \frac{L/m}{r^2}\frac{d}{d\theta}(r).$$
   Similarly
   $$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{L/m}{r^2}\frac{d}{d\theta}(r)\right) = \frac{L/m}{r^2}\frac{d}{d\theta}\left(\frac{L/m}{r^2}\frac{d}{d\theta}(r)\right)$$
   - Second, we make another change of variables and let $u(\theta) = \frac{1}{r(\theta)}$. Using this change, and the derivative formula above, the main differential equation for r becomes
   $$\frac{d^2u}{d\theta^2} = -u + \frac{GM}{(L/m)^2}.$$
   - It is easy to verify that $u = A\cos(\theta) + \frac{GM}{(L/m)^2}$ solves this equation for any value of A.
   - Thus
   $$r = \frac{\frac{(L/m)^2}{GM}}{1 + A\frac{(L/m)^2}{GM}\cos(\theta)}.$$
   But this is the equation of an ellipse in polar coordinates!

2. The area ΔA swept out in a very small time period Δt is approximately
   $$\Delta A \approx \frac{1}{2} r(t)^2 \theta'(t)\Delta t.$$
   Integrate this and use the conservation of angular momentum L to obtain the second of Kepler's laws. In fact, you should get that $\frac{dA}{dt} = \frac{L/m}{2}$.

3. From Kepler's second law and the formula for the area of an ellipse, we have that the period T of the orbit satisfies
   $$T = \frac{2\pi ab}{L/m},$$
   where a, b are the major and minor axes of the ellipse. Using the polar form of the ellipse above, and doing a bunch of gymnastics with equations about ellipses, you should find that
   $$T^2 = \frac{4\pi^2 a^3}{GM}.$$