Chapter 3

Energy and motion

3.1 Newton's Second Law and systems of equations

In this section of the course we study a types of equations motivated by examples from physics. Our starting point is Newton's description of motion in terms of forces, captured by the famous formula $F = ma$. We want to view this formula as a differential equation. To do this we assume that we have a particle of mass m that is moving in one direction. We let $x(t)$ be the function that gives us the location of the particle at time t. The velocity v is given by $v(t) = x'(t)$ and the acceleration a is given by $a(t) = v'(t)$. Thus the formula $F = ma$ can be written in two different ways:

1. We can write $F = ma$ as the *second order differential equation*

$$m\frac{d^2x}{dt^2} = F.$$

2. We can write $F = ma$ as the *first order system*

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = \frac{1}{m}F.$$

Both of these two perspectives capture the same mathematical content.

In order to get a differential equation that we can analyze we need to know F. In general, figuring out precisely what forces we need/want to consider is a dark art best left to the physics department. However, there are an handful of situations where it is relatively easy to understand the formulas. There are three types of situations

- the force F depends only on the location x, in which case $F = F(x)$;
- the force F depends on both the location x and the velocity v, in which case $F = F(x, v)$; and

- the force F depends on the location x, the velocity v, and the time t, in which case $F = F(x, v, t)$.

In this chapter of the course we work with the first two cases; in the next chapter we consider one class of examples of the third case.

Let's give some concrete examples. Our first example is an important model of oscillations.

Example 3.1.1. *Suppose our particle having mass m is attached to a spring in such a way that the object will remain at rest if placed at $x = 0$, but that the object will feel a force from the spring if displaced from location $x = 0$. Hooke's Law is the postulation that the force of the spring upon the mass is proportional to the displacement. Mathematically, we express this assertion as*

$$F_{spring} = -kx,$$

where $k > 0$ is known as the "spring constant." This expression for the force gives us the differential equation

$$m\frac{d^2x}{dt^2} = -kx,$$

which is known as the ***simple harmonic oscillator (SHO)****.*

We can write the SHO equation as the following first-order system:

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{k}{m}x.$$

Notice that this is a linear system! Thus we can also write SHO as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix},$$

It is easy to find the eigenvalues for this matrix. They are

$$\lambda_{\pm} = \pm i\sqrt{\frac{k}{m}}.$$

Thus the equilibrium at $(x, v) = (0, 0)$ is a center, and solutions consist of periodic orbits in the phase plane.

In the homework you use eigenvectors to construct the general solution.

Example 3.1.2. *(work needed)*

A simple model of damping is $F_{damping} = -bv$. Thus the damped simple harmonic oscillator equation is

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}.$$

The corresponding first order system is linear and takes the form

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix},$$

What are the eigenvalues for this matrix?

Example 3.1.3. *If a simple harmonic oscillator is subject to both damping and to an external, time-dependent forcing $f(t)$ then we obtain the equation*

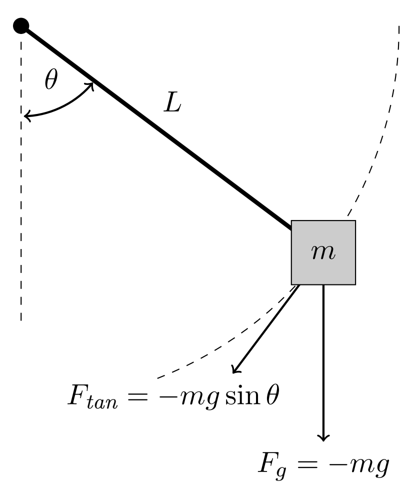
$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + f(t).$$

Equations of this sort are studied in the next chapter of this course.

Here we consider a classic example, the nonlinear pendulum.

Example 3.1.4. *We suppose that a weight of mass m is affixed to the end of a rigid and lightweight rod having length L. The other end of the rod is fixed in such a way that the rod can rotate freely in a vertical plane. We describe the motion of the mass in terms of the angle that the rod makes with the downward direction.*

We assume that the only force acting on the mass is due to gravitation; near the surface of the earth we use the formula that this force has magnitude $F_g = -mg$ in the vertical direction. Here $g \approx 10$ m/s^2 is Newton's "little" gravitational constant. Since the rod is rigid, the only motion of the mass permitted is tangental to the circle of radius L centered at the fixed endpoint of the rod. The tangential component of the gravitational force, which we denote F_{tan}, is given by $F_{tan} = -mg\sin\theta$. This situation is described by the following figure:



We now derive a differential equation for $\theta(t)$ using Newton's second law. The tangential acceleration of the mass is given by

$$\frac{d^2}{dt^2}\left[L\theta\right].$$

Thus the tangential component of the relation $ma = F$ becomes

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta,$$

which we rewrite as

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta.$$

We can express this as the first order system

$$\frac{d\theta}{dt} = \omega \qquad \frac{d\omega}{dt} = -\frac{g}{L}\sin\theta.$$

Exercise 3.1.1. *Use eigenvectors to compute the general solution to the simple harmonic oscillator equation. Deduce from your general solution that*

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

for some constants A and B. We call this the general solution for x.

Draw the graph (in the tx plane) of a typical solution. What information about the graph does the quantity $\sqrt{\frac{k}{m}}$ tell us? This quantity is called the frequency of the oscillator and is often given the symbol ω.

Exercise 3.1.2. *Consider the damped simple harmonic oscillator*

$$\frac{d^2x}{dt^2} = -4x + b\frac{dx}{dt}.$$

1. *Write this as a linear, first-order system that involves a matrix.*
2. *Find the eigenvalues of the matrix. They should involve the letter b.*
3. *Explain how we know that when b is "small" solutions oscillate. This is called the underdamped scenario. What does a typical solution $x(t)$ look like in this situation?*
4. *Explain how we know that when b is "large" the solutions do not oscillate. This is called the overdamped scenario. What does a typical solution $x(t)$ look like in this situation?*
5. *For this particular equation, what is the threshold between "small" and "large"?*

Exercise 3.1.3. *Use technology to draw the vector field phase plot for the nonlinear pendulum. Then find all of the equilibrium points of the system, marking them on your diagram. How many equilibrium points are there? How can you physically interpret the equilibrium points?*

Exercise 3.1.4. *Find an equilibrium point of the nonlinear pendulum system which, based on the phase diagram, is a center.*

- *Linearize the system about this equilibrium and show that indeed it is a center by finding the eigenvalues.*
- *Compare the linearized system to that of the simple harmonic oscillator.*
- *Solve the simple harmonic oscillator equation in order to find a relationship between* L *and the frequency of the oscillations.*
- *Suppose you want to make a pendulum clock, where the pendulum oscillates with a period of one second. How long should* L *be?*

Exercise 3.1.5. *Find an equilibrium point of the nonlinear pendulum system which, based on the phase portrait, seems to be unstable. Linearize the system about this equilibrium and show that indeed the equilibrium point is unstable.*

Exercise 3.1.6. *Suppose we have a system where the force depends only on position and velocity, so that* $F = F(x, v)$.

1. *Show that* $v = 0$ *is always a nullcline in the* xv *phase space diagram.*
2. *I claim that in the phase space diagram, motion is always to the right in top half of the phase diagram (above the* x *axis) and to the left in the lower half. Explain why this is true.*

3.2 Energy and Hamilton's equations

> *There is a fact... called the* conservation of energy*. It states that there is a certain quantity, which we call energy, that does not change in the manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same.*
>
> –Richard Feynman, *Lectures on Physics*, Chapter 4

Suppose we have a particle moving along the x axis. We describe the motion of the particle with a function $x(t)$ that tells us the location at time t. Physicists have devised two quantities that describe the energy of such a particle:

- The ***kinetic energy K*** is the energy associated to the motion of the particle. The kinetic energy is given by the formula

  $$K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2.$$

  If we set $v = \frac{dx}{dt}$, then the formula for the kinetic energy can be written as either

  $$K = \frac{1}{2}mv^2.$$

- The ***potential energy V*** is the energy associated to the particle being at a particular location. Consequently, we typically write the potential energy as function $V(x)$, where x is the location of the particle.

The ***total energy H*** is given by

$$H = K + V = \frac{1}{2}mv^2 + V(x). \tag{3.2.1}$$

Note that the quantity H is really a function of depends on both $x(t)$ and $v(t)$. If we want to explicitly indicate this, we can write

$$H[x(t), v(t)] = \frac{1}{2}mv(t)^2 + V(x(t)),$$

or simply

$$H[x, v] = \frac{1}{2}mv^2 + V(x).$$

Since H depends on x, and x depends on t, the energy H can also be viewed as a function of t. We say that the energy H is ***conserved for function x*** if H is a constant function in time. We can mathematically express this ***conservation of energy*** by

$$\frac{d}{dt}H = 0.$$

Using the chain rule, we see

$$\frac{d}{dt}H = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt}. \tag{3.2.2}$$

Thus we see that energy is conserved if x and v satisfy

$$\frac{dx}{dt} = \frac{1}{m}\frac{\partial H}{\partial v} \qquad \frac{dv}{dt} = -\frac{1}{m}\frac{\partial H}{\partial x}.$$

These equations are known as ***Hamilton's equations***.[1] Equations that can be written in this form for some function H are called ***Hamiltonian systems***.

Hamilton's equations ensure that H is conserved, without making any assumptions on the form of H. When H takes the form $H = \frac{1}{2}mv^2 + V(x)$ then Hamilton's equations become

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{m}V'(x). \tag{3.2.3}$$

We now compare this to the first-order form of Newton's formula $F = ma$, which we wrote as

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = \frac{1}{m}F.$$

Thus we see that Newton's formula agrees with Hamilton's formula if the force F depends only on x and is given by $F(x) = -V'(x)$ for some function V. Such a function V is called the ***potential function*** for the force F. If we can find such a potential function, then we can construct a formula for the total energy H, and we know that this energy is conserved by solutions to the equation.

Example 3.2.1. *Consider the simple harmonic oscillator equation*

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{k}{m}x.$$

Let's look for a potential function V for this equation. Such a function must satisfy

$$-\frac{1}{m}V'(x) = -\frac{k}{m}x.$$

[1] A note for those who have seen Hamilton's equations in a physics class: Here are are using the velocity v, rather than the momentum $p = mv$, as our second variable. This is why we have the factors of $1/m$ in our equations.

Thus we see that one possible potential function is $V(x) = \frac{1}{2}kx^2$. *Using this potential function, we have that the energy*

$$H = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

is conserved by solutions to the equation.

To see what we mean by this, recall that

$$x(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$$

is a solution. For this solution we have

$$H = \frac{1}{2}m\left(-\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)\right)^2 + \frac{1}{2}k\cos\left(\sqrt{\frac{k}{m}}t\right)^2 = \cdots = k,$$

which is constant.

The previous example shows how to start with a differential equation and construct a conserved energy. The next example shows how to start with an energy and construct a differential equation for which that energy is conserved.

Example 3.2.2. *Consider the potential function* $V(x) = -(x-2)^2(x-5)$. *Find the Hamiltonian system for which the energy*

$$H = \frac{1}{2}v^2 - (x-2)^2(x-5)$$

is conserved.

Activity 3.2.1. *Consider the second order equation*

$$\frac{d^2x}{dt^2} = -x^2.$$

Show that the corresponding first-order system is a Hamiltonian system. What is a conserved energy for this system?

Notice that when completing Activity 3.2.1, you actually had some freedom when constructing the potential function. If we are given a differential equation that can be written in the form of a Hamiltonian system, then the potential function is not uniquely determined. We can always add a constant to the potential function and obtain a conserved energy. Mathematically, this is simply a consequence of the fact that the derivative of a constant is zero, which implies that if any particular energy is conserved, then so is that energy plus a constant. In physics, the fact that one can always add a constant to the potential function is in indication that it is the "potential difference" or the relative (rather than absolute) value of the potential that is "physically meaningful."

Thus far we have established that solutions to Hamiltonian systems of the form obey conservation energy. We now show how to use this conservation of energy in order to plot the trajectory of solutions in phase space.

First, we note that because the energy H depends only on v and x, we can determine the energy of a solution from the initial conditions

$$x(0) = x_0 \qquad v(0) = v_0.$$

In particular, the energy of a solution with these initial condition is

$$H = \frac{1}{2}mv_0^2 + V(x_0).$$

Once we know the energy H of a given solution, then the conservation of energy implies that at all times we have

$$H = \frac{1}{2}mv^2 + V(x).$$

This implicitly tells us the trajectory that the solution takes through phase space.

Example 3.2.3. *Suppose x, v is a solution to the simple harmonic oscillator system with initial conditions*

$$x(0) = x_0, \qquad v(0) = v_0.$$

The energy of this solution is

$$H = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2. \tag{3.2.4}$$

At all future times, the solution satisfies

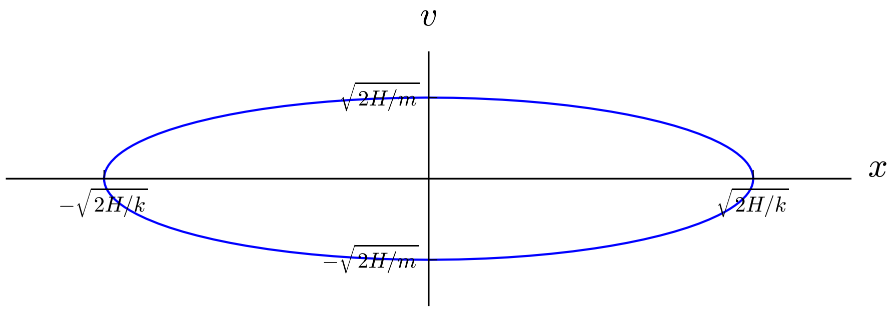
$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2,$$

which we rewrite as

$$v^2 + \frac{k}{m}x^2 = v_0^2 + \frac{k}{m}x_0^2.$$

Thus the solution traverses an ellipse in the phase plane.

With H as in (3.2.4), the trajectory is as shown in the following diagram:



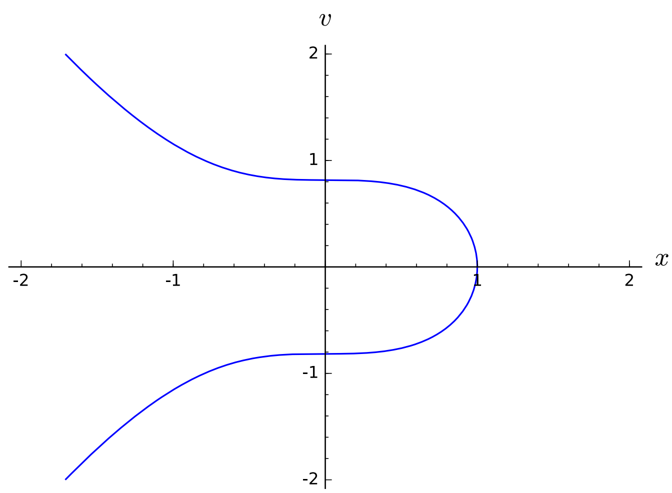
The spatial maximal extent of the solution is $x = \pm\sqrt{2H/k}$*, while the maximal velocities achieved by the solution are* $v = \pm\sqrt{2H/m}$*.*

Activity 3.2.2. *Suppose* x *is the solution to the equation in Activity 3.2.1 having initial conditions*

$$x(0) = 1, \qquad x'(0) = 0.$$

What is the trajectory that the solution will follow in phase space?

You should find that the trajectory looks like the following:



We conclude this section with an example of a system that is *not* a Hamiltonian system.

Example 3.2.4. *Oscillator equations with a non-zero damping term are not Hamiltonian systems. To see this, recall that the first-order system corresponding to the generic oscillator equation is*

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -bv - kx.$$

Since the right side of the second equation is not a function of x *alone it does not take the form of a Hamiltonian system. Thus there cannot be a conserved energy for the system.*

Exercise 3.2.1. *From previous work we know that the general solution to the simple harmonic oscillator equation is*

$$x(t) = \alpha \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + \beta \sin\left(\sqrt{\tfrac{k}{m}}\, t\right).$$

Show by direct computation that the quantity

$$H = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} k x^2$$

is constant, thereby confirming that H is conserved.

Exercise 3.2.2. *Suppose that $x(t)$ is a solution to the initial value problem*

$$4\frac{d^2x}{dt^2} + 9x = 0, \qquad x(0) = 1, \qquad x'(0) = 2.$$

Describe the trajectory of this solution in phase space. What is the maximal velocity attained by the solution? What is the maximal spatial location?

Exercise 3.2.3. *Consider the differential equation*

$$\frac{d^2x}{dt^2} = x^2.$$

1. *Write the equation in the form of a first-order Hamiltonian system and find the conserved energy.*
2. *Let $x(t)$ be the solution to the initial value problem*

   $$x(0) = 1, \qquad x'(0) = 0.$$

   What is the energy of this solution?
3. *Use conservation of energy to determine the shape of the trajectory that the solution takes in phase space.*

Exercise 3.2.4. *Construct a Hamiltonian system, and an initial condition, for which the corresponding solution traverses the hyperbola*

$$\frac{1}{2}v^2 - \frac{1}{2}x^2 = 50.$$

How many such solutions can you find?

Exercise 3.2.5. *In this problem you study energy for a damped oscillator*

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + x = 0.$$

1. *Write this equation as a first-order system for the variables x and v.*
2. *Let $H = \frac{1}{2}v^2 + \frac{1}{2}x^2$. Supposing that (x, v) is a solution to the first order system, show that $\frac{d}{dt}H = -bv^2$.*
3. *Conclude that for $b > 0$ we have H is decreasing. Give a graphical interpretation of this on the phase portrait. How can one use this fact to draw conclusions about the long-term behavior of solutions?*

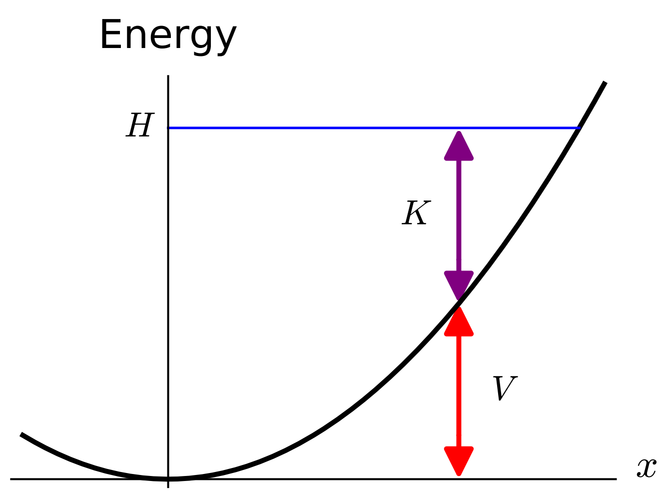
3.3 Energy diagrams

We now introduce a new tool for visualizing solutions to Hamiltonian systems of the form

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -V'(x). \tag{3.3.1}$$

An ***energy diagram*** is a plot of solutions where the horizontal axis is the spatial position x and the vertical axis is the energy H. Since energy is conserved, the trajectory of a solution to (3.3.1) in an energy diagram is along a horizontal line, the height of which is the energy of that solution. At each point along the trajectory, we can decompose the height, that is the energh H, in to two components: the kinetic $K = \frac{1}{2}mv^2$ and the potential energy $V = V(x)$. The amount of potential energy at each location is given by the potential function $V(x)$, and thus we can compute the amount of kinetic energy K by simply subtracting.

Example 3.3.1. *Consider the simple harmonic oscillator, for which the potential function is $V(x) = \frac{1}{2}kx^2$ for some constant k. In the energy diagram below, the solution traverses the blue path at height H. At any given spatial location, the total energy H is the sum of the potential part V and the kinetic part K. The amount of energy that is potential energy is indicated by the potential function $V(x)$, plotted in black. At each location, the vertical "gap" between the potential function and the total amount of energy indicates the amount of kinetic energy K.*

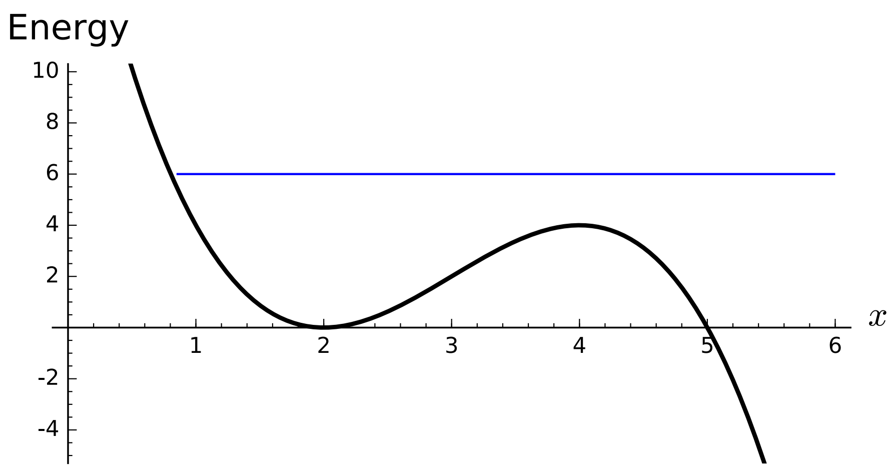


We now make a key observation: the kinetic energy $K = \frac{1}{2}mv^2$ can never be negative. As a consequence, the trajectory of a solution in the energy diagram is always "above" the curve defined by the potential function $V(x)$. Thus a solution will move horizontally across the energy diagram until it reaches the graph of the potential function. At the point where the trajectory reaches this "potential curve" the kinetic energy, and thus the velocity, is zero. Hence the solution instantaneously comes to a halt. The solution, however, does not remain stationary. Consider, for example, the illustration in Example 3.3.1 above. A solution moving to the right at height H reaches the potential curve at a location where $V'(x)$ is positive. Thus at that moment, we deduce from the differential equation (3.3.1) that $v' < 0$. Thus the velocity instantaneously decreases from being zero to being negative, at which point the solution begins to move to the left. In this way we see that solutions "reflect" off the graph of the potential function.

Example 3.3.2. *Consider the potential function* $V(x) = -x(x-2)^2(x-5)$*. The corresponding Hamiltonian system is*

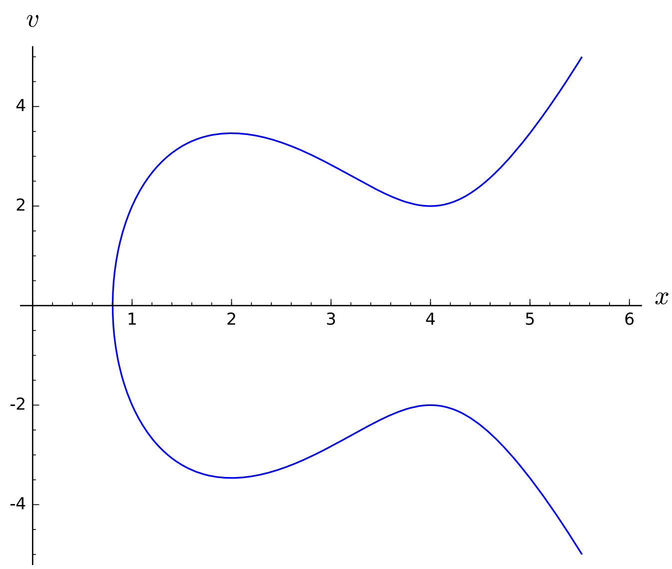
$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = 3(x-2)(x-4). \tag{3.3.2}$$

The energy diagram for this potential, together with the trajectory of a solution having energy $H = 6$ *is the following:*

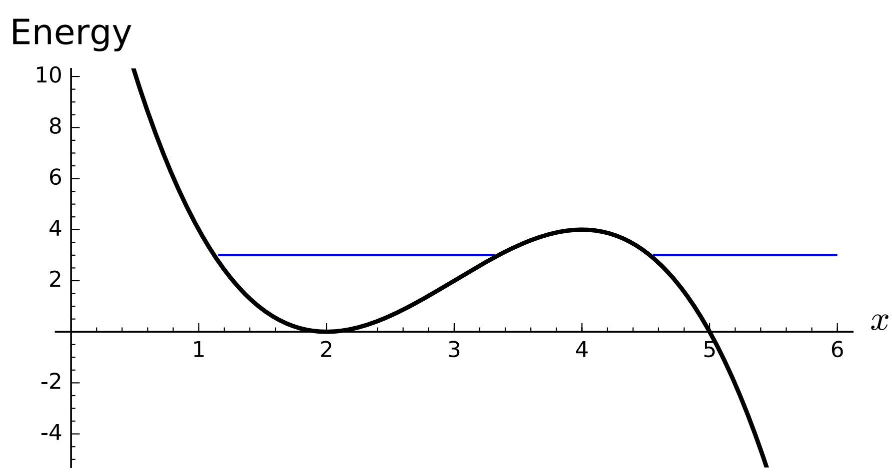


The solution begins at $x = 6$ *and negative* v*. It moves to the left along the line* $H = 6$ *until the trajectory encounters the potential curve at* $x \approx 0.8$*. The solution then reflects off the potential wall and begins to move to the right, still along* $H = 6$*, continuing for all future times.*

We can further understand this solution by plotting the trajectory in the phase plane. Initially, when $x = 6$*, we see that the kinetic energy is large and thus* v *is large in magnitude; since the solution is moving to the left,* v *is negative. As the solution passes over the "bump" at* $x = 4$*, the magnitude of* v *decreases, but then increases again as the solution passes through* $x = 2$*. Then* v *decreases to zero as the solution approaches the potential curve. Following the reflection off the potential curve, the velocity* v *is positive and growing,* *achieving a local maximum at $x = 2$ and then decreases as x approaches 4. Subsequently, v increases without bound. From this we can deduce that the trajectory of this solution in the phase plane is the following:*



Activity 3.3.1. *Consider again the potential $V(x) = -x(x-2)^2(x-5)$. Describe the behavior of two different solutions to (3.3.2), both having with energy $H = 3$, that appear in the following energy diagram.*



Draw the corresponding trajectories in the phase plane.

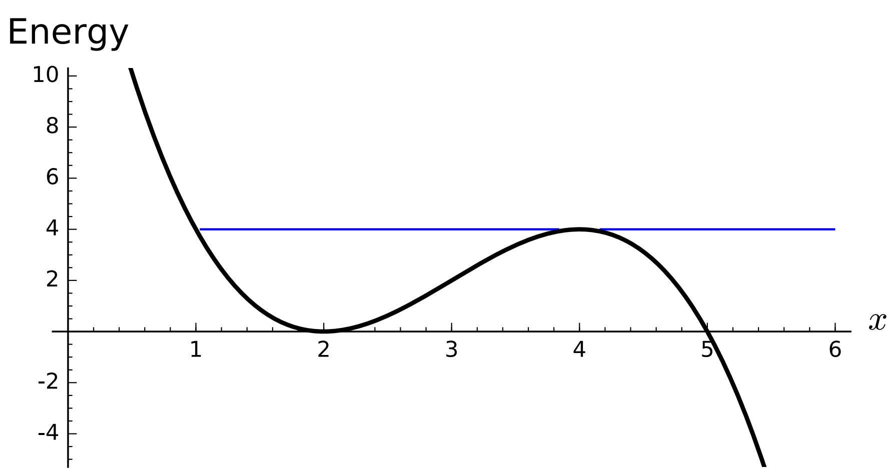
One of the solutions with energy $H = 3$ is a ***bound solution*** *and one is an* ***unbounded solution****. Which is which? Use this example to develop a more general definition of these terms.*

In the previous activity and example, we considered solutions to the system (3.3.2) whose trajectory in the energy diagram either passed well over the "bump" at $x = 4$ or were not high enough to reach close to the top of that bump. However, it is possible that a solution have precisely the right energy to approach the potential curve at a local maximum of the potential function. We now investigate how solutions behave in such a situation.

Suppose that we have a potential function $V(x)$ having a local maximum at $x = x_*$. This means that $V'(x_*) = 0$. Examining the differential equation (3.3.1), we see that $(x, v) = (x_*, 0)$ is an equilibrium solution. In the energy diagram, this solution simply "sits" at the point $(x, H) = (x_*, V(x_*))$.

We now consider the situation where there is another solution x, v to (3.3.1) that has energy $H = V(x_*)$ and whose trajectory in the energy diagram is approaching $(x_*, V(x_*))$. Since we cannot have two different solutions to the equation (3.3.1) meet, we conclude that this second solution x, v approaches the equilibrium point asymptotically, but never reaches it.

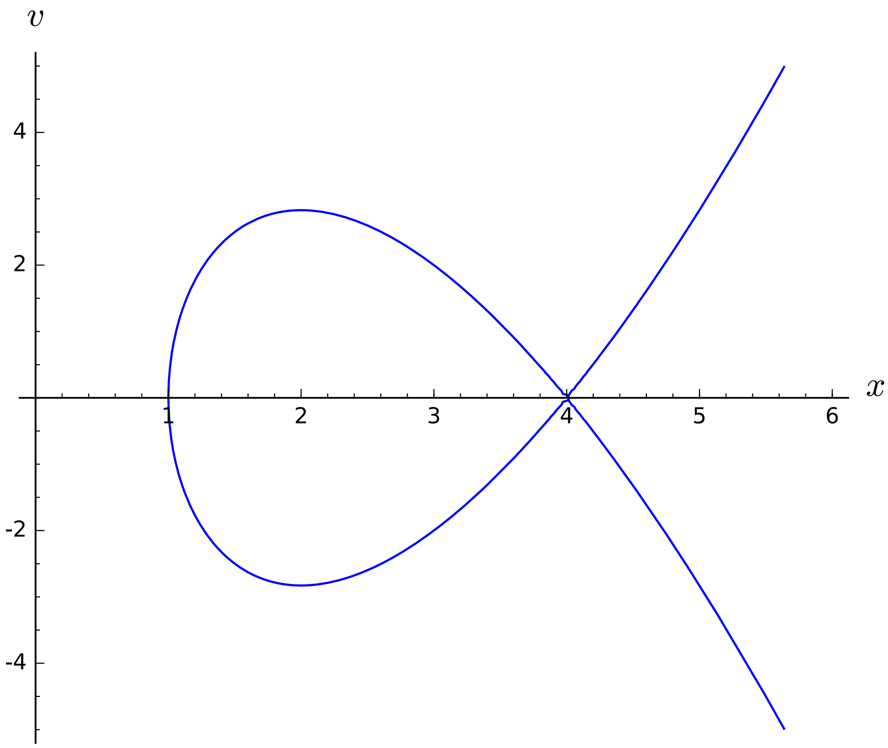
Example 3.3.3. *Consider once more the potential function $V(x) = -x(x-2)^2(x-5)$. Solutions with energy $H = 4$ appear in the energy diagram as follows:*



Those solutions that begin to the left of $x_* = 4$ with negative velocity move to the left until they reflect off the potential curve; then they move to the right, asymptotically approaching $x_* = 4$. Those solutions that initially have $x < 4$ but begin with positive velocity simply asymptote directly to $x_* = 4$.

Solutions to the right of $x_* = 4$ tend towards the equilibrium if their initial velocity is negative, but away from it is their initial velocity is positive.

In the phase diagram, these trajectories are as follows:



We now investigate the equilibrium points for (3.3.1) more systematically. Our first observation is that *all* of the equilibrium points take the form $(x, v) = (x_*, 0)$, where x_* is a root of $V'(x)$. In other words, the equilibrium points of a Hamiltonian system are precisely the critical points of the potential function.

Activity 3.3.2. *Suppose* $(x, v) = (0, x_*)$ *is an equilibrium point for* (**??**). *Show that the linearization of* (3.3.1) *at* $(0, x_*)$ *is given by*

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -V''(x_*) & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}.$$

Using the results of this activity, we see that the eigenvalues of the linearized system are

$$\lambda = \pm\sqrt{-V''(x_*)}.$$

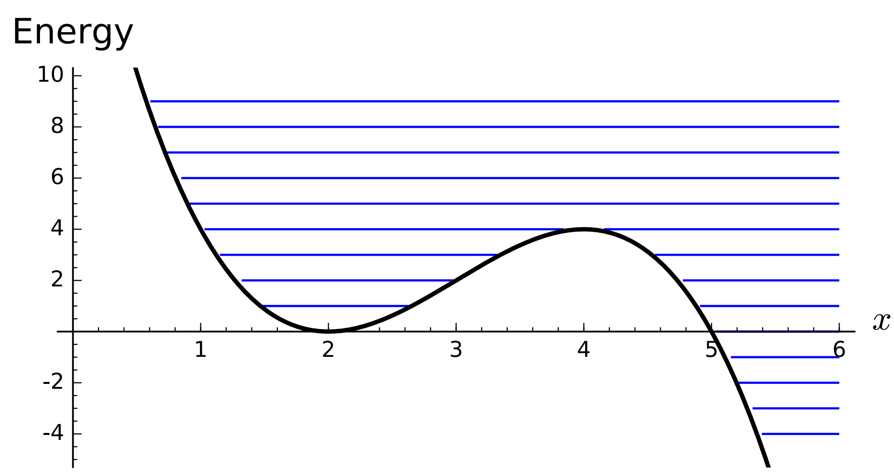
Thus:

- If $V''(x_*) > 0$, then $(x_*, 0)$ is a center-type equilibrium.
- If $V''(x_*) < 0$, then $(x_*, 0)$ is a saddle-type equilibrium.

Of course, if $V''(x_*) = 0$, then we do not learn anything about the stability type of the equilibrium from the linearization. However, it is still the case that if x_* is a local minimum of V then $(x_*, 0)$ is a center and if x_* is a local maximum of V then $(x_*, 0)$ is a saddle. If x_* is a critical point of V but is neither a local minimum nor local maximum, then one obtains a ***degenerate equilibrium***.

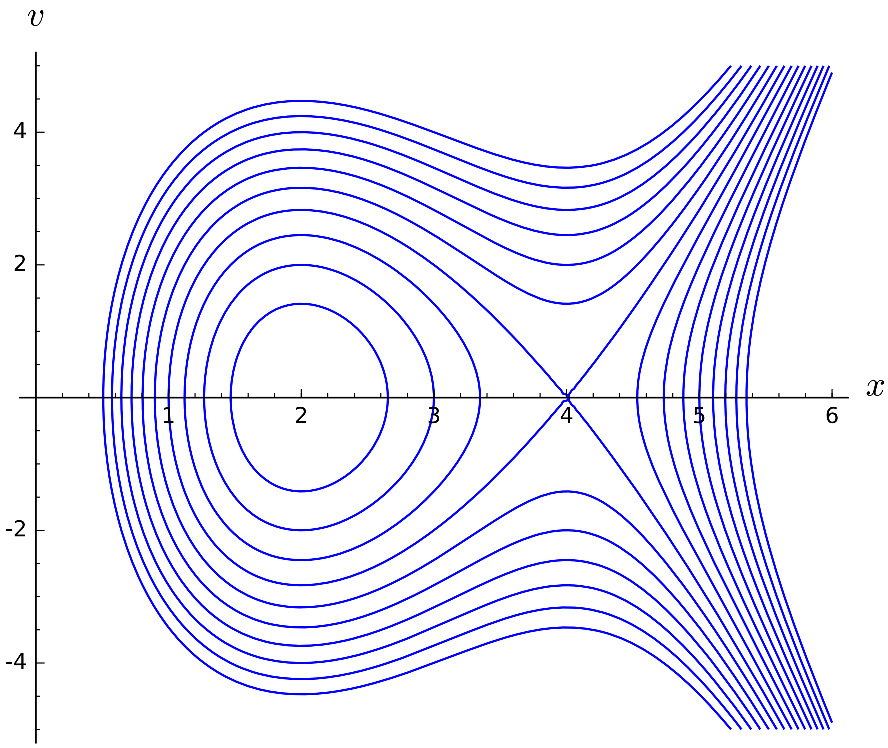
The result of this discussion is that we can entirely understand the solutions to a Hamiltonian system by examining the shape of the potential function. Critical points of the potential function are equilibrium points, and are either saddle-, center-, or degenerate-type. By considering solutions of various energies, we can deduce which initial conditions give rise to bound solutions, and which give rise to unbound solutions. Finally, by tracing each solution through the energy diagram, we can easily deduce the shape of the corresponding solution in the phase plane.

Example 3.3.4. *Consider one final time the potential function $V(x) = -x(x-2)^2(x-5)$. Here we plot the energy diagram with a variety of solution trajectories having different energy levels:*



Notice that we have a center-type equilibrium solution at $x_* = 2$ and a saddle-type equilibrium solution at $x_* = 4$. Solutions with energy $H < 4$ and having $x(0) < 4$ will be bound solutions; these solutions oscillate about the equilibrium $x_* = 2$. All solutions with $H > 4$ or with $H < 4$ but $x(0) > 4$ have $x \to \infty$ as $t \to \pm\infty$. Finally, solutions with $H = 4$ will tend to the equilibrium at $x_* = 4$, unless we have both $x(0) > 4$ and $v(0) > 0$, in which case $x \to \infty$ as $t \to \infty$.

Using this information, we can easily draw the corresponding trajectories in the phase plane:



Exercise 3.3.1. *Suppose we have the forced oscillator*

$$\frac{d^2x}{dt^2} + 9x = 10$$

1. *Write this as a first order system.*
2. *Show that* $H = \frac{1}{2}v^2 + \frac{9}{2}x^2 - 10x$ *is a conserved quantity.*
3. *Draw the energy diagram for the equation.*
4. *Draw the phase portrait for the equation.*
5. *Discuss the long-term behavior of solutions to the system, based on your diagram & portrait.*

Exercise 3.3.2. *Consider the differential equation*

$$\frac{d^2x}{dt^2} = x^3 - x$$

1. *Write the equation as a first-order system.*

2. *Find the potential function V and the conserved energy H.*

3. *Carefully draw the energy diagram. Be sure to include several typical trajectories.*

4. *Carefully draw the phase portrait. Include the paths corresponding to the typical trajectories you put on the energy diagram.*

5. *Give a written description of the types of solutions that exist and their behavior.*

Exercise 3.3.3. *Repeat the steps in Exercise 3.3.2 for the differential equation*

$$\frac{d^2x}{dt^2} = e^{-x} - 1.$$

Exercise 3.3.4. *Consider the potential function $V(x) = \frac{-1}{1+x^2}$.*

1. *Write down the corresponding first-order Hamiltonian system.*

2. *Draw the energy and phase diagrams.*

3. *Describe the behavior of solutions to the system.*

Exercise 3.3.5. *Repeat the previous exercise with the potential function $V(x) = \frac{1}{x} + x$.*

Exercise 3.3.6. *Here you analyze the prototypical case of a degenerate critical point of the potential function.*

- *Work with the potential function $V(x) = x^3$.*

- *Write down the corresponding Hamiltonian system.*

- *Linearize the system at the equilibrium point $(0, 0)$.*

- *Compute the eigenvalues of the matrix for the linearized system, and verify that this situation is indeed "degenerate" in the sense that the linearized system doesn't have two eigensolutions.*

Exercise 3.3.7. *In this exercise you explore the nullclines of Hamiltonian systems.*

1. *First consider the generic Hamiltonian system (3.3.1). Describe the nullclines of this system; your description will likely involve the potential function. How can you deduce what the nullclines will be from the plot of the potential function V?*

2. *Find the nullclines of the system in Exercise 3.3.2.*