Chapter 5

Fundamental Theorems

5.1 Revisiting the 1D Fundamental Theorem of Calculus

Key Ideas.

- *Alternate take on the derivative of a function* $f\colon \mathbb{R} \to \mathbb{R}$.
  - *The function* f *determines a one-dimensional vector field* $\vec{V} = \langle f \rangle = f\partial_x = f\mathbf{i}$.
  - *Let* $B_r = [p_* - r, p_* + r]$ *be the interval of "radius"* r *centered at point* p_*.
  - *The boundary of* B_r *consists of two points* $p_* + r$ *and* $p_* - r$. *The "outward unit normal" to* B_r *is given by*

$$\hat{N} = \begin{cases} \partial_x & \text{at } x = p_* + r \\ -\partial_x & \text{at } x = p_* - r. \end{cases}$$

  - *The outflux of* $\vec{V}$ *across the boundary of* B_r *is*

$$\sum_{\text{boundary points}} \vec{V} \cdot \hat{N} = f(p_* + r) - f(p_* - r).$$

  - *The outflux per unit size is* $\dfrac{f(p_* + r) - f(p_* - r)}{2r}$
  - *Linear approximation gives us* $\dfrac{f(p_* + r) - f(p_* - r)}{2r} = f'(p_*) + \text{error}$, *where error* $\to 0$.
- *Thus the derivative of* f *at* p_* *is the limit of the "effect per unit size" of the function.*
- *The Fundamental Theorem of Calculus can be obtained by summing over small intervals...*
- *Our plan is now to find notions of derivatives, and corresponding fundamental theorems, for various quantities...*

5.2 Fundamental Theorem for Gradients

Key Ideas.

- *Consider a function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *and a path* $P(t) = (x(t), y(t), \dots)$ *defined for* $a \le t \le b$. *Let* $A = P(a)$ *and* $B = P(b)$. *We have* $f \circ P\colon [a, b] \to \mathbb{R}$. *By the 1D Fundamental Theorem of Calculus we have*
$$\int_a^b (f \circ P)'(t)\, dt = f(A) - f(B).$$
- *From the Chain Rule we have*
$$(f \circ P)'(t) = [Df(P(t))]P'(t) = \begin{pmatrix} \frac{\partial f}{\partial x}(P(t)) \\ \frac{\partial f}{\partial y}(P(t)) \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \vdots \end{pmatrix}$$
- *Recall that the* ***gradient*** *of a function* f*:*
$$\operatorname{grad}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots \right\rangle.$$
- *Geometric idea of gradient: The gradient of a function measures the total change of a function. If* $\hat{A}$ *is any unit vector, then* $\operatorname{grad}(f) \cdot \hat{A}$ *tells us the extent to which* f *changes in that direction. (This is the same as the directional derivative.) The gradient vector points in the direction of largest increase, with magnitude equal to the rate of change in that direction.*
- *Differentiation rules:*
  - *Scaling rule:* $\operatorname{grad}(cf) = c \operatorname{grad}(f)$ *if* c *is a constant*
  - *Addition rule:* $\operatorname{grad}(f + g) = \operatorname{grad}(f) + \operatorname{grad}(g)$
  - *Product rule:* $\operatorname{grad} fg = \operatorname{grad}(f)g + f \operatorname{grad}(g)$
- **Fundamental Theorem of Calculus for gradients***: Suppose* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is smooth*[1] *and that* C *is a parametrized path from point* A *to point* B*. Then*
$$\int_C \operatorname{grad}(f) \cdot \hat{T}\, ds = f(B) - f(A).$$

Exercises.

1. Compute the gradient for the following functions:
   (a) $f(x, y) = x^2 + y^2$
   (b) $f(x, y) = x^2 - y^2$
   (c) $f(x, y) = \ln(x^2 + y^2)$

[1] In this course ***smooth*** means that the object is continuous and that derivatrives of the object are also continuous.

(d) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

2. Can you find a function f so that...

   (a) ...$\text{grad}(f) = \langle 2, 3\rangle$?

   (b) ...$\text{grad}(f) = \langle x, y\rangle$?

   (c) ...$\text{grad}(f) = \langle y, x\rangle$?

   (d) ...$\text{grad}(f) = \langle y, -x\rangle$?

3. Let C be the path traveling counter-clockwise along the unit circle from $(1, 0)$ to $(-1, 0)$. Let L be the path traveling along the x axis from $(1, 0)$ to $(-1, 0)$.

   (a) Find parametrizations for C and L.

   (b) Compute $\int_C \langle x, y\rangle \cdot \hat{T}\, ds$ by direct methods.

   (c) Compute $\int_L \langle x, y\rangle \cdot \hat{T}\, ds$ by direct methods.

   (d) Find a function f such that $\text{grad}(f) = \langle x, y\rangle$.

   (e) Compute $\int_C \langle x, y\rangle \cdot \hat{T}\, ds$ using the Fundamental Theorem for Gradients.

   (f) Compute $\int_L \langle x, y\rangle \cdot \hat{T}\, ds$ using the Fundamental Theorem for Gradients.

   (g) Do your computations agree? Explain why/why not.

4. Let C be the path traveling counter-clockwise along the unit circle from $(1, 0)$ to $(-1, 0)$. Let $\widetilde{C}$ be the path traveling clockwise along the unit circle from $(1, 0)$ to $(-1, 0)$.

   (a) Find parametrizations of C and $\widetilde{C}$.

   (b) Compute $\int_C \langle y, -x\rangle \cdot \hat{T}\, ds$ by direct methods.

   (c) Compute $\int_{\widetilde{C}} \langle y, -x\rangle \cdot \hat{T}\, ds$ by direct methods.

   (d) Do your computations agree? Explain why/why not.

5.3 Divergence of a vector field

Key Ideas.

- *Geometric idea of divergence: The divergence of vector field* $\vec{V}$ *at the point* P_* *measures the expansion of* $\vec{V}$ *at* P_*
- *Definition of divergence using limits: Let* B_r *be the solid ball of radius* r *centered at point* P_* *and* S_r *be the boundary of* B_r.

$$\operatorname{div}\vec{V}(P_*) = \lim_{r\to 0}\frac{1}{|B_r|}\iint_{S_r}\vec{V}\cdot\hat{N}\,dA \qquad \textit{and/or} \qquad \operatorname{div}\vec{V}(P_*) = \lim_{r\to 0}\frac{1}{|B_r|}\int_{S_r}\vec{V}\cdot\hat{N}\,ds$$

- *Computation in Cartesian coordinates: Suppose* $\vec{V} = \langle V_x, V_y, \dots\rangle$. *Then*

$$\operatorname{div}\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \dots.$$

- *Differentiation rules:*
  - *Scaling rule:* $\operatorname{div}(c\vec{V}) = c\operatorname{div}\vec{V}$ *if* c *is a constant*
  - *Addition rule:* $\operatorname{div}(\vec{V}+\vec{W}) = \operatorname{div}\vec{V} + \operatorname{div}\vec{W}$
  - *Product rule:* $\operatorname{div}(f\vec{V}) = \operatorname{grad}(f)\cdot\vec{V} + f\operatorname{div}\vec{V}$ *if* f *is a function*

Exercises.

1. For each vector field do the following: make a sketch of the vector field by hand, describe in words what the vector field does, compute the divergence of the vector field, comment on how your computation fits with your earlier description.

   (a) $\vec{V} = \langle x, y\rangle$

   (b) $\vec{V} = \langle -y, x\rangle$

   (c) $\vec{V} = \langle 3, -2\rangle$

   (d) $\vec{V} = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\rangle$

   (e) $\vec{V} = \langle x-y, x+y\rangle$

   (f) $\vec{V} = \langle x+y, x-y\rangle$

   (g) $\vec{V} = \langle y^2, y^2\rangle$

2. Suppose $f(x,y) = x^2 - y^2$ and $\vec{V} = \langle x+y, x+2y\rangle$.

   (a) Compute $f\vec{V}$ and then compute $\operatorname{div}(f\vec{V})$.

   (b) Use the product rule to compute $\operatorname{div}(f\vec{V})$.

   (c) Prove the product rule formula for divergence.

3. The ***Laplacian*** of a function f is defined by $\operatorname{div}(\operatorname{grad} f)$. Compute the Laplacian of the following functions.

   (a) $f(x, y) = 2x - 3y$

   (b) $f(x, y) = x^2 + y^2$

   (c) $f(x, y) = x^2 - y^2$

   (d) $f(x, y) = \cos(x)\sin(y)$

   (e) $f(x, y) = \frac{1}{2}\ln(x^2 + y^2)$

   (f) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

   What property of the function does the Laplacian seem to measure?

4. Consider the vector field given in spherical coordinates by

$$\vec{V} = \frac{1}{r^2}\partial_r = \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle$$

   and concentric spheres S_r of radius r centered at the origin.

   (a) Compute the outward flux $\iint_{S_r} \vec{V} \cdot \hat{N}\, dA$ using the definition of the flux integral;

   (b) Find the flux rate

$$\lim_{r \to 0} \frac{\iint_{S_r} \vec{V} \cdot \hat{N}\, dA}{\operatorname{Volume}(B_r)}$$

   where B_r denotes the ball enclosed by S_r.

   (c) Compute the divergence of the vector field $\vec{V}$.

   (d) What is the outward flux rate of $\vec{V}$ at the origin? What about anywhere other than the origin?

5. (Optional Challenge Problem) In this problem you consider the possibility of defining divergence of a vector field $\vec{V}(x, y)$ using a square, rather than a ball. Fill in the boxes in the argument below.

   (a) For simplicity we work in two dimensions. Write $\vec{V}(x, y) = \langle V_x(x, y), V_y(x, y) \rangle$. Fix a point (x_*, y_*). For small x and y we have

$$\vec{V}(x, y) \approx \vec{V}(x_*, y_*) + (x - x_*) \left\langle \frac{\partial V_x}{\partial x}(x_*, y_*), \frac{\partial V_y}{\partial x}(x_*, y_*) \right\rangle$$
$$+ (y - y_*) \left\langle \boxed{}, \boxed{} \right\rangle$$

   (b) Let B_r denote the interior of the square in $\mathbb{R}^2$ having corners at $(x_* + r, y_* + r)$, $(x_* - r, y_* + r)$, $(x_* - r, y_* - r)$, $(x_* + r, y_* - r)$. Let S_r be the boundary of B_r, oriented counter-clockwise.

Draw a picture showing B_r and S_r.

The area of B_r is $\boxed{\quad\quad}$

(c) The outward unit normal $\hat{N}$ along S_r is given by

$$\hat{N} = \begin{cases} \langle 1, 0 \rangle & \text{when } x = x_* + r, \\ \boxed{\quad\quad} & \text{when } y = y_* + r, \\ \boxed{\quad\quad} & \text{when } \boxed{\quad\quad}, \\ \boxed{\quad\quad} & \text{when } \boxed{\quad\quad}, \end{cases}$$

(d) The integral $\int_{S_r} \vec{V} \cdot \hat{N}\, ds$ can be approximated by

$$\int_{t=-r}^{r} \vec{V}(x_* + r, y_* + t) \cdot \langle 1, 0 \rangle \, dt + \int_{t=r}^{-r} \boxed{\quad\quad\quad\quad} \cdot \langle 0, 1 \rangle \, dt$$

$$+ \int_{t=r}^{-r} \boxed{\quad\quad\quad\quad} \, dt + \boxed{\quad\quad\quad\quad\quad\quad}.$$

(e) Using the approximation from the first part of this problem we we find that

$$\int_{S_r} \vec{V} \cdot \hat{N}\, ds \approx \int_{t=-r}^{r} \left(\vec{V}(x_*, y_*) + r \frac{\partial V_x}{\partial x}(x_*, y_*) + t \frac{\partial V_x}{\partial y}(x_*, y_*) \right) dt$$

$$+ \int_{t=r}^{-r} \left(\vec{V}(x_*, y_*) + t \boxed{\quad\quad\quad} + r \boxed{\quad\quad\quad} \right) dt$$

$$- \int_{t=r}^{-r} \left(\vec{V}(x_*, y_*) + r \boxed{\quad\quad\quad} - t \boxed{\quad\quad\quad} \right) dt$$

$$- \int_{t=-r}^{r} \boxed{\quad\quad\quad\quad\quad\quad\quad\quad} \, dt$$

(f) The previous integral simplifies to

$$\int_{S_r} \vec{V} \cdot \hat{N}\, ds \approx \boxed{\quad\quad} \left(\frac{\partial V_x}{\partial x}(x_*, y_*) + \boxed{\quad\quad\quad} \right).$$

Thus

$$\lim_{r\to 0} \frac{\int_{S_r} \vec{V}\cdot\hat{N}\,ds}{\boxed{}} = \frac{\partial V_x}{\partial x}(x_*, y_*) + \boxed{}$$

and we are done!

5.4 Divergence Theorems

Key Ideas.

- ***Two-dimensional Divergence Theorem***. *Let D be a bounded domain in $\mathbb{R}^2$, let C be the boundary of D, and let $\hat{N}$ be the outward-pointing unit normal along C. Then for any smooth vector field $\vec{V}$ defined in D we have*

$$\iint_D \operatorname{div} \vec{V}\, dA = \int_C \vec{V} \cdot \hat{N}\, ds.$$

The two-dimensional divergence theorem is equivalent to ***Green's Theorem****. (However, Green's theorem is usually written in a different way – see the next section.)*

- ***Three-dimensional Divergence Theorem***. *Let D be a bounded domain in $\mathbb{R}^3$, let S be the boundary of D, and let $\hat{N}$ be the outward-pointing unit normal along S. Then for any smooth vector field $\vec{V}$ defined in D we have*

$$\iiint_D \operatorname{div} \vec{V}\, dV = \int_S \vec{V} \cdot \hat{N}\, dA.$$

The three-dimensional divergence theorem is sometimes called ***Gauss' Theorem****.*

- *Justification: divide D in to many small domains and use the definition of divergence.*

- *Note: The vector field $\vec{V}$ and its divergence need to be defined in all of D.*

- *Important application: If $\operatorname{div} \vec{V} = 0$ in a region D whose boundary has two parts: S_1 and S_2 then*

$$\iint_{S_1} \vec{V} \cdot \hat{N}\, dA + \iint_{S_2} \vec{V} \cdot \hat{N}\, dA = 0.$$

(A similar formula holds if D is a region in $\mathbb{R}^2$.)

Vector fields whose divergence is zero are called called ***divergence-free*** *or* ***incompressible****.*

Exercises.

1. Consider the circles C_0, C_1, C_2, C_3, and the outward pointing vector fields $\hat{N}$ as in Figure 1.

   Suppose $\vec{V}(x, y)$ is a vector field defined (and smooth) throughout the xy-plane. Furthermore, suppose that $\vec{V}$ has the following outward fluxes:

$$\int_{C_0} \vec{V} \cdot \hat{N}\, ds = 1, \quad \int_{C_1} \vec{V} \cdot \hat{N}\, ds = 3, \quad \int_{C_2} \vec{V} \cdot \hat{N}\, ds = -2, \quad \int_{C_3} \vec{V} \cdot \hat{N}\, ds = 5.$$

   Use Green's Theorem to evaluate $\iint_D \operatorname{div}(\vec{V})\, dA$ where D denotes

   (a) The area inside C_0 but outside C_2;

   (b) The area inside C_0 but outside both C_1 and C_3.

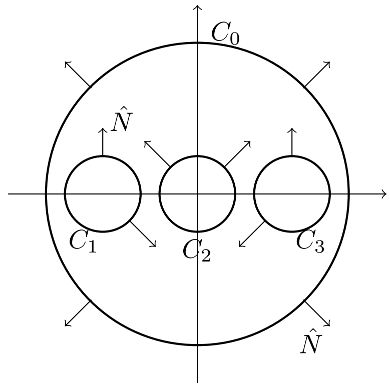


Figure 5.1: The figure for homework problem 1.

2. Evaluate the outward flux integrals $\int_C \vec{V} \cdot \hat{N}\, ds$ for the following choices of the vector fields $\vec{V}$ and the curves C. You are expected to use the divergence theorem.

   (a) $\vec{V}(x, y) = \langle 3y, x \rangle$, while C is the counter-clockwise unit circle centered at the origin.

   (b) $\vec{V}(x, y) = \langle x^2 + xy + y^2,\, x^2 - xy + y^2 \rangle$, while C is the boundary of the rectangle with vertices at $(0, 0)$, $(1, 0)$, $(1, 2)$, $(0, 2)$.

   (c) $\vec{V}(x, y) = \langle x, y \rangle$, while C is the boundary of the triangle with vertices at $(-1, 0)$, $(0, 1)$ and $(1, 0)$.

3. Use Gauss' Theorem to evaluate the following flux integrals.

   (a) The outward flux of $\vec{V}(x, y, z) = \langle x^3 + y^3, y^3 + z^3, z^3 + x^3 \rangle$ across the unit sphere centered at the origin;

   (b) The inward flux of $\vec{V}(x, y, z) = \langle xy, yz, xz \rangle$ across the bowl of the elliptic paraboloid $z = x^2 + y^2$

      i. up to and including the lid at $z = 4$;

      ii. up to but *not* including the lid at $z = 4$;

   (c) The inward flux of the vector field $\vec{V}(x, y, z) = \langle x^2 + x,\ y^2 + y,\ -z^2 - z \rangle$ across the cone $z = \sqrt{x^2 + y^2}$

      i. up to and including the lid at $z = 2$;

      ii. up to but *not* including the lid at $z = 2$;

   (d) The outward flux of $\vec{V}(x, y, z) = \langle x, y, z \rangle$ across the cube $-1 \le x, y, z \le 1$.

5.5 Scalar curl; 2D Curl Theorem; Green's Theorem

Key Ideas.

- *Geometric idea of scalar curl: The curl of a 2D vector field $\vec{V}$ at a point P_* measures the counter-clockwise rotational work done by $\vec{V}$ at that point.*
- *Definition of scalar curl using limits: Let B_r be the solid disk of radius r centered at P_* and let S_r be the circular boundary of B_r and let $\hat{T}$ be the counter-clockwise unit tangent along S_r. The scalar curl of vectorfield $\vec{V}$ is defined by*
$$\operatorname{curl}(\vec{V}) = \lim_{r\to 0} \frac{1}{|B_r|} \int_{C_r} \vec{V}\cdot\hat{T}\,ds.$$
- *Computation in Cartesian coordinates: Suppose $\vec{V} = \langle V_x, V_y\rangle$. Then* $\operatorname{curl}(\vec{V}) = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$.
Computation using determinants: $\operatorname{curl}\vec{V} = \det\begin{pmatrix} \frac{\partial}{\partial x} & V_x \\ \frac{\partial}{\partial y} & V_y \end{pmatrix}$
- *Differentiation rules:*
  - *Scaling rule:* $\operatorname{curl}(c\vec{V}) = c\operatorname{curl}\vec{V}$
  - *Addition rule:* $\operatorname{curl}(\vec{V}+\vec{W}) = \operatorname{curl}\vec{V} + \operatorname{curl}\vec{W}$
  - *Product rule:* $\operatorname{curl}(f\vec{V}) = \det\begin{pmatrix} \frac{\partial f}{\partial x} & V_x \\ \frac{\partial f}{\partial y} & V_y \end{pmatrix} + f\operatorname{curl}\vec{V}$.
- ***The two-dimensional Curl Theorem**: Let D be a bounded domain in $\mathbb{R}^2$, let C be the boundary of D, and let $\hat{T}$ be the counter-clockwise oriented tangent vector along C. For any smooth vector field $\vec{V}$ defined in D we have*
$$\iint_D \operatorname{curl}\vec{V}\,dA = \int_C \vec{V}\cdot\hat{T}\,ds.$$
- *Justification of the 2D Curl Theorem: Divide D in to many small squares...*
- *Suppose that $\vec{V} = \langle P, Q\rangle$. Then* $\operatorname{curl}(\vec{V})\,dA = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$. *Suppose also that C is parametrized by the functions $x(t), y(t)$. Since $dx = x'(t)$ and $dy = y'(t)$ we can write $\hat{T}\,ds = \langle dx, dy\rangle$ and thus $\vec{V}\cdot\hat{T}\,ds = Pdx + Qdy$. Thus the 2D Curl Theorem can be written*
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_C Pdx + Qdy.$$
*This formula is known as **Green's Theorem**.*
- *Suppose that $\vec{W} = \langle Q, -P\rangle$. Then* $\operatorname{div}(\vec{W})dA = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$. *With C parametrized by the functions $x(t), y(t)$ we have $dx = x'(t)$ and $dy = y'(t)$ as above. Thus $\hat{N}\,ds = \langle dy, -dx\rangle$ and the 2D divergence theorem for $\vec{W}$ becomes*
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_C Pdx + Qdy.$$
Green's Theorem encapsulates both the 2D Curl Theorem and the 2D Divergence Theorem.

Exercises.

1. Draw a picture of each vector field. Use the picture to estimate the scalar curl. Then compute the scalar curl and compare.

   (a) $\vec{V} = \langle x, y \rangle$

   (b) $\vec{V} = \langle y, x \rangle$

   (c) $\vec{V} = \langle y, -x \rangle$

   (d) $\vec{V} = \left\langle \dfrac{y}{x^2+y^2}, \dfrac{-x}{x^2+y^2} \right\rangle$

2. Use Green's Theorem to evaluate the circulation integrals $\int_C \vec{V} \cdot \hat{T}\, ds$ for the following choices of $\vec{V}$ and C.

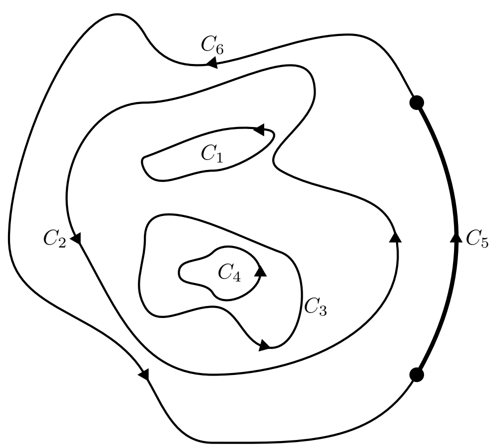
   (a) $\vec{V}(x,y) = \langle 3y, x \rangle$, while C is the counter-clockwise unit circle centered at the origin.

   (b) $\vec{V}(x,y) = \langle x^2 + xy + y^2,\, x^2 - xy + y^2 \rangle$, while C is the boundary of the rectangle with vertices at $(0,0)$, $(1,0)$, $(1,2)$, $(0,2)$ oriented counter-clockwise.

   (c) $\vec{V}(x,y) = \langle 3x^2y + y^3,\, -x^3 - 3xy^2 \rangle$, while C is the boundary of the unit circle centered at the origin oriented counter-clockwise.

   (d) $\vec{V}(x,y) = \langle x, y \rangle$, while C is the boundary of the triangle with vertices at $(-1,0)$, $(0,1)$ and $(1,0)$ oriented clock-wise.

3. Consider the configuration on the following diagram. Note that C_5 is the bold path with distinct starting and ending points, and that C_6 denotes the "left-over".



Suppose $\vec{V}(x,y)$ is a vector field defined (and smooth) throughout the xy-plane. Furthermore, suppose the following:

$$\int_{C_1} \vec{V} \cdot \hat{T}\, ds = -1,\quad \int_{C_2} \vec{V} \cdot \hat{T}\, ds = 1,\quad \int_{C_3} \vec{V} \cdot \hat{T}\, ds = -2,\quad \int_{C_4} \vec{V} \cdot \hat{T}\, ds = 2,\quad \int_{C_5} \vec{V} \cdot \hat{T}\, ds = -3,$$

while $\mathrm{curl}(\vec{V})$ vanishes outside of C_2. Use Green's Theorem to evaluate

(a) $\iint_D \mathrm{curl}(\vec{V})\,dA$ where D denotes the area inside C_3 but outside C_4;

(b) $\iint_D \mathrm{curl}(\vec{V})\,dA$ where D denotes the area inside C_2 but outside C_1;

(c) $\iint_D \mathrm{curl}(\vec{V})\,dA$ where D denotes the area inside C_2 but outside C_4;

(d) $\iint_D \mathrm{curl}(\vec{V})\,dA$ where D denotes the area inside C_2 but outside both C_1 and C_4;

(e) $\int_{C_6} \vec{V} \cdot \hat{T}\,ds$.

4. Let S_r denote the square of side $2r$ given by:

$$a - r \leq x \leq a + r,$$
$$b - r \leq y \leq b + r.$$

Thus the center of the square is the point (a, b). Let C_r be the boundary of that square, parametrized counter-clockwise. Let $\vec{V}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be some vector field defined near (a, b). Show that

$$\lim_{r \to 0} \frac{\int_{C_r} \vec{V} \cdot \hat{T}\,ds}{\mathrm{Area}(S_r)} = \frac{\partial Q}{\partial x}(a, b) - \frac{\partial P}{\partial y}(a, b).$$

Discuss the intuitive meaning of this result in a sentence or so.

5.6 3D curl; Stokes' Theorem

Key Ideas.

- *Geometric idea of 3D curl: The 3D curl (usually just called "curl") describes the total rotation of a vector field in* $\mathbb{R}^3$*. If* $\hat{A}$ *is any unit vector, then* $\mathrm{curl}(\vec{V}) \cdot \hat{A}$ *represents the extent to which* $\vec{V}$ *is rotating about the axis defined by* $\hat{A}$*. (This is equivalent to saying that* $\mathrm{curl}(\vec{V}) \cdot \hat{A}$ *represents the extent to which* $\vec{V}$ *is rotating in the plane that has* $\hat{A}$ *as its unit normal.) Thus* $\mathrm{curl}(\vec{V})$ *points in the direction about which there is the most rotation, which the magnitude equal to the amount of rotation and the sign determined by the "right hand" positivity convention.*

- *Definition of curl: Suppose* $\vec{V} = \langle V_x, V_y, V_z \rangle$*. Then*

$$\mathrm{curl}(\vec{V}) = \langle \mathrm{curl}(\langle V_y, V_z \rangle), -\mathrm{curl}(\langle V_x, V_z \rangle), \mathrm{curl}(\langle V_x, V_y \rangle) \rangle.$$

  The minus sign is due to $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

- *Computation in Cartesian coordinates: Suppose* $\vec{V} = \langle V_x, V_y, V_z \rangle$*. Then*

$$\mathrm{curl}(\vec{V}) = \left\langle \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, -\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \right), \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\rangle$$

  Computation using determinants:

$$\mathrm{curl}(\vec{V}) = \det \begin{pmatrix} \mathbf{i} & \frac{\partial}{\partial x} & V_x \\ \mathbf{j} & \frac{\partial}{\partial y} & V_y \\ \mathbf{k} & \frac{\partial}{\partial z} & V_z \end{pmatrix}$$

- *Differentiation rules:*
  - *Scaling rule:* $\mathrm{curl}(c\vec{V}) = c\,\mathrm{curl}(\vec{V})$
  - *Addition rule:* $\mathrm{curl}(\vec{V} + \vec{W}) = \mathrm{curl}(\vec{V}) + \mathrm{curl}(\vec{W})$
  - *Product rule:* $\mathrm{curl}(f\vec{V}) = \mathrm{grad}(f) \times \vec{V} + f\,\mathrm{curl}(\vec{V})$.

- *The 3D Curl Theorem. Suppose* S *is a (two-sided) surface in* $\mathbb{R}^3$ *and that the curve* C *is the boundary of* S*. We orient* S *and* C *by choosing unit normal vector* $\hat{N}$ *on* S *and unit tangent vector* $\hat{T}$ *on* C *so that* $\hat{T} \times \hat{N}$ *is the outward normal to* S*. Then for any smooth vector field* $\vec{V}$ *we have*

$$\iint_S \mathrm{curl}(\vec{V}) \cdot \hat{N}\, dA = \int_C \vec{V} \cdot \hat{T}\, ds.$$

  *The 3D Curl Theorem is called **Stokes' Theorem** (though Stokes' theorem is more general).*

- *Justification: Divide* D *in to many squares ...*

Exercises.

1. Compute the curl of the following vector fields:
   (a) $\vec{V} = \langle x, y, z \rangle$
   (b) $\vec{V} = \langle z, x, y \rangle$
   (c) $\vec{V} = \langle 0, z, -y \rangle$
   (d) $\vec{V} = \langle x + y, y + z, z + x \rangle$

2. Let $\vec{V}(x, y, z) = \langle yz, -xz, xy \rangle$.
   (a) What is the circulation rate of $\vec{V}$ around the vector $\hat{N} = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$ at the following points:
      i. $(1, 0, 0)$;
      ii. $(0, 1, 0)$;
      iii. $(0, 0, -1)$.
   (b) Repeat the above for the circulation around the vector $\hat{N} = \langle 0, 1, 0 \rangle$.
   (c) Around which vector (or in other words, in which plane) does $\vec{V}$ circulate the most? What is the maximum circulation rate?

3. Consider the configuration in Figure 5.2. Note: the surface S_1 is just the "volcanos" up to C_3, the surface S_2 is just the "head" and the surface S_3 is just the bowl on the bottom. The normal vector field $\hat{N}$ is outward pointing, as indicated.

   Suppose $\vec{V}(x, y, z)$ is a vector field defined (and smooth) throughout the xyz-space. Use Stokes' Theorem to evaluate
   (a) $\int_{C_3} \vec{V} \cdot \hat{T}\, ds$ if it is known that $\iint_{S_3} \text{curl}(\vec{V}) \cdot \hat{N}\, dA = 2$;
   (b) $\iint_{S_2} \text{curl}(\vec{V}) \cdot \hat{N}\, dA$ if it is known that $\iint_{S_3} \text{curl}(\vec{V}) \cdot \hat{N}\, dA = 1$;
   (c) $\iint_{S_1} \text{curl}(\vec{V}) \cdot \hat{N}\, dA$ if it is known that $\int_{C_1} \vec{V} \cdot \hat{T}\, ds = \int_{C_2} \vec{V} \cdot \hat{T}\, ds = \int_{C_3} \vec{V} \cdot \hat{T}\, ds = 3$;
   (d) $\int_{C_2} \vec{V} \cdot \hat{T}\, ds$ if it is known that $\iint_{S_1} \text{curl}(\vec{V}) \cdot \hat{N}\, dA = \iint_{S_2} \text{curl}(\vec{V}) \cdot \hat{N}\, dA = 3$ and $\int_{C_1} \vec{V} \cdot \hat{T}\, ds = -2$.

4. Let $\vec{V}(x, y, z) = \langle -y, x, 0 \rangle$, let S be the dome of the upper unit hemisphere centered at the origin, and let C be the equator of the hemisphere in the xy-plane, oriented counter-clockwise. Compute:
   (a) The outward flux $\iint_S \text{curl}(\vec{V}) \cdot \hat{N}\, dA$ using the definition of the flux integral;
   (b) The circulation $\int_C \vec{V} \cdot \hat{T}\, ds$ using the definition of the circulation integral.
   (c) In a sentence or so, comment on the relationship between the answers to parts (a) and (b), and the Stokes' Theorem.

5. Use Stokes' Theorem to evaluate the circulation integrals $\int_C \vec{V} \cdot \hat{T}\, ds$ for the following choices of $\vec{V}$ and C.

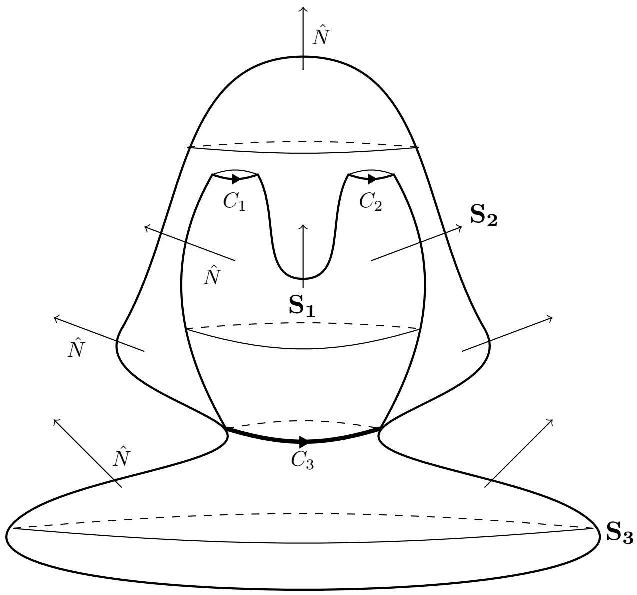


Figure 5.2: The figure for Problem 3.

(a) $\vec{V}(x, y, z) = \langle z, x, y \rangle$ while C is the contour of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in that order.

(b) $\vec{V}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ while C is the contour of a triangle with vertices $(0, 0, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ traversed in that order.

(c) $\vec{V}(x, y, z) = \langle 0, -z, y \rangle$ while C is the North 45-th parallel of the unit sphere centered at the origin, oriented counterclockwise when viewed from the above.

5.7 The infamous "del" notation

Much of the physics and math literature uses the formal object "del" given by

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

to concisely express ideas in vector calculus. It is important to keep in mind that while this object uses vector notation, it is not a vector – it's just a formal object that allows us to write down various formulas in a convenient way.

Here is a short summary of facts and formulas that can be written using the del object.

- Derivatives (gradient, divergence, curl):

$$\text{grad}(f) = \vec{\nabla} f \qquad \text{div}(\vec{V}) = \vec{\nabla} \cdot \vec{V} \qquad \text{curl}(\vec{V}) = \vec{\nabla} \times \vec{V}$$

- Product rules for gradient, divergence, curl:

$$\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g \qquad \vec{\nabla} \cdot (f\vec{V}) = \vec{\nabla} f \cdot \vec{V} + f\vec{\nabla} \cdot \vec{V} \qquad \vec{\nabla} \times (f\vec{V}) = \vec{\nabla} f \times \vec{V} + f\vec{\nabla} \times \vec{V}$$

- Divergence Theorem:

$$\iiint_D \vec{\nabla} \cdot \vec{V}\, dV = \iint_C \vec{V} \cdot \hat{N}\, dA$$

- Curl Theorem:

$$\iint_S (\vec{\nabla} \times \vec{V}) \cdot \hat{N}\, dA = \int_C \vec{V} \cdot \hat{T}\, ds.$$

- The Laplacian of a function has two common notations:

$$\text{div}(\text{grad}(u)) = \Delta u \qquad \text{div}(\text{grad}(u)) = \vec{\nabla}^2 u.$$

Exercises.

1. Let u be a function.
   (a) Translate the quantity $\vec{\nabla} \times (\vec{\nabla} u)$ into the language of grad/div/curl.
   (b) Show by brute force computation that $\vec{\nabla} \times (\vec{\nabla} u) = 0$ for all functions u.
2. Let $\vec{V}$ be a vector field.
   (a) Translate the quantity $\text{div}(\text{curl}(\vec{V}))$ into del notation.
   (b) Show by brute force computation that $\text{div}(\text{curl}(\vec{V})) = 0$ for all vector fields $\vec{V}$.
3. Show by brute force computation that $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = -(\vec{\nabla} \times \vec{A}) \cdot \vec{B} + \vec{A} \cdot (\vec{\nabla} \times \vec{B})$.