Chapter 4

Integration

4.1 Line Integrals

Key Ideas.

- *Recall construction of the integral:* $\int_a^b f(t)dt = \lim_{\Delta t \to 0} \sum_k f(t_k)\Delta t$, *where* $t_k = a + k\Delta t$.

  Interpretation:

  – *Suppose* F *is an anti-derivative of* f.
  – *Linear approximation* $F(t_k + \Delta t) \approx F(t_k) + f(t_k)\Delta t$
  – *Fundamental Theorem of Calculus:* $\int_a^b f(t)dt = F(b) - F(a)$

- *Arc-length integrals:*

  – *Consider a parametrized path* $P\colon [a,b] \to \mathbb{R}^n$.
  – *Linear approximation:* $\|P(t_k + \Delta t) - P(t_k)\| \approx \|DP(t_k)\|\Delta t$
  – *Definition of arc-length integral*

$$\int_a^b ds = \lim_{\Delta t \to 0} \sum_k \|DP(t_k)\|\Delta t = \int_a^b \|DP(t)\|dt.$$

  – *Arc-length element* $ds = \|DP(t)\|dt$. *Alternatively,* $ds = \|\partial_t\|dt$.

- *Integrating a function along a path:*

  – *Suppose that we have a path* C *parametrized by* $P\colon [a,b] \to \mathbb{R}^n$, *and that we also have a function* $f\colon \mathbb{R}^n \to \mathbb{R}$.
  – *The integral of* f *along* C *is* $\int_C f\, ds = \int_a^b f(P(t))\, \|DP(t)\|\, dt$.
  – *Average value of* f *along* C *is* $\frac{1}{|C|}\int_C f\, ds$, *where* $|C|$ *the length of the curve.*

Exercises.

1. Find the formula for the line element ds, and use it to compute the total length of the given curve. If the parametric equations are not explicitly given, please parametrize the curve first.

   (a) The helicoidal spiral given by

   $$\begin{aligned} x(t) &= e^t \cos(3t) \\ y(t) &= e^t \sin(3t) \qquad -1 \le t \le 1 \\ z(t) &= 3e^t \end{aligned}$$

   (b) The circle in the xy-plane with center at $(3, 2)$ and radius 4;

   (c) The North 45^{th}-parallel of the unit sphere centered at the origin;

   (d) The $30°$-parallel of the southern unit hemisphere (centered at the origin).

2. Let C be the unit circle centered at the point $(2, 0)$. Parametrize C and compute:

   (a) $\int_C x\, ds$;

   (b) $\int_C y\, ds$.

3. Let C be the straight line segment from $(0, 1, 1)$ to $(1, 0, 2)$. Parametrize C and compute

   $$\int_C xyz\, ds.$$

4. Let C be the path consisting of a semi-circle joining $(1, 0)$ to $(-1, 0)$ in the upper half-plane, and the diameter joining $(-1, 0)$ and $(1, 0)$. Compute

   $$\int_C (x^2 + y^2)\, ds.$$

5. A thin wire in the shape of circle of radius 2 inches is centered at the origin of the coordinate plane. The linear weight density of the wire at the location with coordinates (x, y) is $\varrho(x, y) = 2 + xy$ ounces per inch. (Assume that inches are used as the unit of spatial measurement throughout the problem.)

   (a) What is the total weight of the wire?

   (b) What is the average weight density of the wire?

6. What is the average of $\varrho(x, y, z) = x^2$ along the North 45th-parallel of the unit sphere centered at the origin?

4.2 The Fubini Theorem

Portrait of Guido Fubini
public domain image acquired from:
https://en.wikipedia.org/wiki/Guido_Fubini

Key Ideas.

- *Let R be a region in $\mathbb{R}^2$ and suppose $f\colon R \to \mathbb{R}$. The definition of the integral of f over the region R is*

$$\iint_R f\,dA = \sum_{i,j} f(x_i, y_j)\Delta x \Delta y$$

- *The Fubini Theorem tells us that we can compute using iterated integrals:*

$$\iint_R f\,dA = \int_*^* \left(\int_*^* f(x,y)\,dy\right)dx = \int_*^* \left(\int_*^* f(x,y)\,dx\right)dy$$

  The limits of integration depend on the region R.

- *The analogous statements hold for regions R inside $\mathbb{R}^3$*

- *Area and volume:*
  - *If R is a region in $\mathbb{R}^2$, then $|R| = \iint_R 1 dA$ is the area of R.*
  - *If R is a region in $\mathbb{R}^3$, then $|R| = \iiint_R 1 dV$ is the volume of R.*

- *The average value of f over a region R is either*

$$\frac{1}{|R|}\iint f\,dA \qquad or \qquad \frac{1}{|R|}\iiint f\,dV$$

  as appropriate.

Exercises.

1. For each part, make a sketch of the region R and compute $\iint_R f\,dA$ using the Fubini Theorem. (You might want to set up the integrals both ways and see which is easier to compute.)

   (a) R is the rectangle $-3 \le x \le 2$, $0 \le y \le 4$ and $f(x,y) = x^2y - xy^2$.

   (b) R is the rectangle $0 \le x \le 1$, $0 \le y \le 1$ and $f(x,y) = xe^{-x-y}$.

   (c) R is the entire Cartesian plane, and
   $$f(x,y) = \frac{1}{(1+x^2)(1+y^2)}.$$

   (d) R is the unit disk centered at the origin and $f(x,y) = 2x + 3y$.

   (e) R is the triangle enclosed by $y = |x|$ and $y = 1$, and $f(x,y) = x^2 + y^2$.

   (f) R is the region enclosed by the parabolas $3y = x^2$ and $3x = y^2$, and $f(x,y) = 1$.

   (g) R is the region outside the unit circle $x^2 + y^2 = 1$ but inside the square given by $-1 \le x, y \le 1$, and $f(x,y) = y^{-2}$.

2. A thin 2 ft $\times$ 1 ft metal sheet covers the rectangle $[-1,1] \times [0,1]$ of the coordinate plane. The weight density of the sheet at the location with coordinates (x,y) is $\varrho(x,y) = x^2 + y^2$ lbs per square foot. (Whenever in doubt in this problem, assume that feet are used as the unit of spatial measurement.)

   (a) What is the total weight of the metal sheet?

   (b) What is the average weight density of the metal sheet?

3. For each part, make a sketch of the region Ω and compute $\iiint_\Omega f\,dV$ using the Fubini Theorem.

   (a) Ω is the box $0 \le x, y, z \le 1$ and $f(x,y,z) = \sqrt{xyz}$.

   (b) Ω is the entire first octant (where $x, y, z \ge 0$) and $f(x,y,z) = e^{-x-y-z}$.

4. Make a sketch of Ω and compute $\iiint_\Omega f(x,y,z)\,dV$, where Ω and f are given as follows. You are expected to use the Fubini Theorem. Also, you may want to set up the integrals in at least two different ways (both according to the Fubini Theorem) and then choose the easier one to complete.

   (a) Ω is the volume between the xy-plane and the graph of the paraboloid $z = x^2 + y^2$ over the domain $-1 \le x, y \le 1$, and $f(x,y,z) = z$.

   (b) Ω is the volume between $z = x^2 + y^2$ and the plane $z = 1$, and $f(x,y,z) = x^3$.

5. (a) Use integration to compute the average value of the function $f(x,y) = xy$ over the standard unit disk.

   (b) The average value of the function $f(x,y) = xy$ over the unit disk, which you just computed, could have be predicted before you computed the integral. Figure out how. Then apply what you learned to the following.

      i. Find the average value of the function $f(x,y) = x^3 + y^3$ over the unit disk.

      ii. Find the average value of the function $f(x,y) = 1 + x^3 + y^3$ over the unit disk.

4.3 Integration Practice

Integration over a triangle

Let R be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, $(0,3)$. Let $f(x,y) = x^2 + y^2$.

- We want to compute $\iint_R f\,dA$ in two different ways: horizontal slicing and vertical slicing.
- First, convince yourself that the diagonal of the triangle is the line $\frac{x}{2} + \frac{y}{3} = 1$.

Horizontal slicing

- Draw a picture.
- We have one slice for each y value $0 \le y \le 3$.
- The left end of each slice is at $x = 0$. The right end of each slice is at $x = 2 - \frac{2}{3}y$.
- Thus
$$\iint_R f\,dA = \int_{y=0}^{y=3} \left(\int_{x=0}^{x=2-\frac{2}{3}y} f(x,y)\,dx \right) dy.$$
- Compute:
$$\begin{aligned}\iint_R f\,dA &= \int_{y=0}^{y=3} \left(\int_{x=0}^{x=2-\frac{2}{3}y} (x^2+y^2)\,dx \right) dy \\ &= \int_0^3 \left(\left[\frac{1}{3}x^3 + y^2 x \right]_{x=0}^{x=2-\frac{2}{3}y} \right) dy \\ &= \int_0^3 \left(\frac{1}{3}\left(2 - \frac{2}{3}y\right)^3 + y^2\left(2 - \frac{2}{3}y\right) \right) dy \\ &\vdots \\ &= \frac{13}{2}\end{aligned}$$

Vertical slicing

- Draw a picture
- We have once slice for each x value $0 \le x \le 2$
- The lower end of each slice is at $y = 0$. The upper end of each slice is at $y = 3 - \frac{3}{2}x$.
- Thus
$$\iint_R f\,dA = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3-\frac{3}{2}x} f(x,y)\,dy \right) dx$$

- Compute
$$\iint_R f\,dA = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3-\frac{3}{2}x} (x^2+y^2)\,dy \right) dx$$
$$= \int_0^2 \left(\left[x^2y + \frac{1}{3}y^3 \right]_{y=0}^{y=3-\frac{3}{2}x} \right) dx$$
$$= \int_0^2 \left(x^2\left(3-\frac{3}{2}x\right) + \frac{1}{3}\left(3-\frac{3}{2}x\right)^3 \right) dx$$

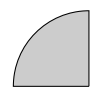
Activity

Have students work with their "friendly neighbors" on this problem. After about 5 minutes, start writing the solution on the board. After 10 minutes, discuss the solution with the whole class.

Let R be the region that is inside the unit circle and is in the second quadrant. Let $f(x,y) = x^2+y^2$.

1. Draw a picture of the region R. What equations that describe the borders of the region?

   Solution.

   The region is bounded by $x = 0$, $y = 0$, and $x^2 + y^2 = 1$. Here is a picture:  □

2. Construct an integral that computes $\iint_R f\,dA$ using vertical slicing. You do not need to actually compute the integral, just do the setup.

   Solution.
$$\iint_R f\,dA = \int_{x=-1}^{x=0} \left(\int_{y=0}^{y=\sqrt{1-x^2}} (x^2+y^2)\,dy \right) dx$$
□

3. Construct an integral that computes $\iint_R f\,dA$ using horizontal slicing. You do not need to actually compute the integral, just do the setup.

   Solution.
$$\iint_R f\,dA = \int_{y=0}^{y=1} \left(\int_{x=-\sqrt{1-y^2}}^{x=0} (x^2+y^2)\,dx \right) dy$$
□

More examples

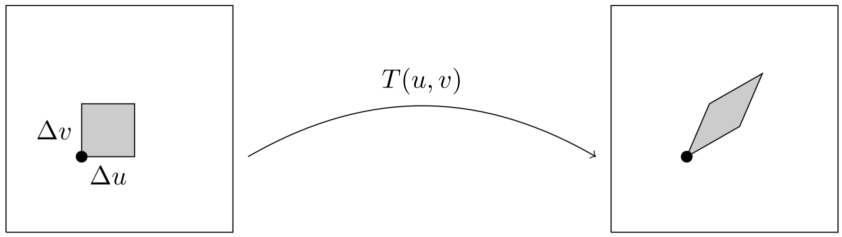
As time permits, work on these examples.

1. Let R be the region bounded by the parabola $y = x^2$ and the line $y = 4-3x$. Let $f(x,y) = xy$.
   - Discuss two different ways to compute $\iint_R f\,dA$
   - Decide which one is easier. . . and compute the integral using the easier way.
2. Let Ω be the region bounded by the parabolic bowl $z = x^2 + y^2$ and the plane $z = 1$. Let f be given by $f(x,y,z) = xyz$. Construct, but do not compute, the integral $\iiint_\Omega f\,dV$.

4.4 The area element and coordinate changes

Key Ideas.

- *Consider a coordinate transformation* $T\colon \mathbb{R}^2 \to \mathbb{R}^2$



- *The "coordinate area element" in uv space is transformed to the "physical area element"* dA *in xy space. There are two ways to compute the physical area element:*

$$dA = |\det(DT)|\, du\, dv \qquad \textit{and} \qquad dA = \sqrt{\det\begin{pmatrix} \partial_u \cdot \partial_u & \partial_u \cdot \partial_v \\ \partial_v \cdot \partial_u & \partial_v \cdot \partial_v \end{pmatrix}}\, du\, dv$$

- *An important special case is polar coordinates* $(x,y) = (r\cos\theta, r\sin\theta)$*. We compute* $dA = r\, dr\, d\theta$.
  - *Use polar coordinates when the region of integration has rotational symmetry.*
  - *There is a classic trick that uses polar coordinates to compute* $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.
- *We can use modified polar coordinates for elliptical regions. The generic ellipse*

$$\left(\frac{x - x_*}{a}\right)^2 + \left(\frac{y - y_*}{b}\right)^2 = 1$$

is parametrized by $x = x_* + au\cos(v)$ *and* $y = y_* + bu\sin(v)$*. We have* $dA = ab\, u\, du\, dv$.

Exercises.

1. Integrate the following functions over the following regions. *All disks and circles are centered at the origin.*

   (a) $f(x,y) = e^{-\frac{x^2+y^2}{2}}$ over the unit disk;

   (b) $f(x,y) = 2\sqrt{9 - x^2 - y^2}$ over the disk of radius 3;

   (c) $f(x,y) = x + y$ over the region in the first quadrant between the circle of radius 1 and the circle of radius 2;

   (d) $f(x,y) = x^2$ over the region above the graph of $y = |x|$, outside the unit circle, but inside the circle of radius 2;

   (e) $f(x,y) = \frac{1}{1+x^2+y^2}$ over the entire Cartesian plane.

2. Use modified polar coordinates to compute the following:
   (a) $\iint_R (x+y)\,dA$ where R is the unit disk centered at $(2,2)$;
   (b) The area of the ellipse $4x^2+9y^2=1$;
   (c) $\iint_R (x^2+y^2)\,dA$ where R is the area inside the ellipse $x^2+9y^2=4$.

3. Construct "diagonal coordinates" that parametrize the square region R whose corners are $(0,0)$, $(1,1)$, $(-1,1)$, $(0,2)$. Use your parametrization to compute $\iint_R y^2\,dA$.

4. (a) Follow the argument from class to show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.
   (b) Based on the above evaluate the following:
      i. $\int_0^{\infty} e^{-x^2}\,dx$;
      ii. $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$.

4.5 Integration in cylindrical and spherical coordinates

Key Ideas.

- *Suppose $(x, y, z) = T(u, v, w)$ is a coordinate transformation $\mathbb{R}^3 \to \mathbb{R}^3$. Then the physical volume element dV is given by*

$$dV = |\det(DT)|\, du\, dv\, dw \qquad \textit{and} \qquad dv = \sqrt{\det \begin{pmatrix} \partial_u \cdot \partial_u & \partial_u \cdot \partial_v & \partial_u \cdot \partial_w \\ \partial_v \cdot \partial_u & \partial_v \cdot \partial_v & \partial_v \cdot \partial_w \\ \partial_w \cdot \partial_u & \partial_w \cdot \partial_v & \partial_w \cdot \partial_w \end{pmatrix}}\, du\, dv\, dw$$

- *The special case of cylindrical coordinates is important. We compute $dV = r\, dr\, d\theta\, dz$.*
- *The special case of spherical coordinates is important. We compute $dV = r^2 \sin\phi\, dr\, d\theta\, d\phi$.*
- *We can also use modified cylindrical and spherical coordinates.*

Exercises.

1. Use cylindrical coordinates to integrate the following functions over the cylinder of radius 1, centered along the z-axis between the planes $z = -1$ and $z = 1$.
   (a) $f(x, y, z) = x^2 + y^2 + z^2$;
   (b) $f(x, y, z) = xyz$.
2. Use cylindrical coordinates to integrate the function $f(x, y, z) = x^2 + y^2 + z^2$ over the region located within the cone $z = \sqrt{x^2 + y^2}$ but under the plane $z = 1$.
3. Use triple integrals and cylindrical coordinates to compute the following volumes:
   (a) Volume contained inside the sphere of radius R.
   (b) Volume contained between the surfaces $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$.
   (c) Volume of the region bounded by $x^2 + y^2 - z^2 = 1$ and the planes $z = \pm 1$.
   (d) Volume of the donut whose overall shape is obtained by the disk $(x - 3)^2 + y^2 \leq 4$ rotating about the y-axis.
4. Use modified cylindrical coordinates to compute the volume contained between $z = x^2 + 4y^2$ and $z = 9$.
5. Consider the transformation

$$(x, y, z) = T(r, v, w) = (u\cos(v)\cos(w), u\sin(v)\cos(w), u\sin(w))$$

   on the domain $u \geq 0$, $-\pi \leq v \leq \pi$, $-\frac{\pi}{2} \leq w \leq \frac{\pi}{2}$.
   (a) Compute the Jacobi matrix of T and identify the coordinate tangent vector fields ∂_u, ∂_v, ∂_w in the xyz-space.
   (b) Compute $\partial_u \cdot \partial_u, \partial_u \cdot \partial_v, ..., \partial_w \cdot \partial_w$.

(c) Compute the volume element dV in (u, v, w)-coordinates in two different ways.

6. Use spherical coordinates to compute:

   (a) The total weight of the ice cream and the cone in the overall shape of

   $$z \geq \sqrt{x^2 + y^2}, \; x^2 + y^2 + z^2 \leq 1$$

   if the weight density is given by $\varrho(x, y, z) = z$. (Suppress the units in this problem.)

   (b) $\iiint_\Omega x \; dV$ where Ω is the volume enclosed by the front unit hemisphere

   $$x \geq 0, \; x^2 + y^2 + z^2 \leq 1.$$

   (c) Average value of the function $f(x, y, z) = x^2 + y^2$ over the ball $x^2 + y^2 + z^2 \leq R^2$.

7. Compute the volume of the ellipsoid $x^2 + 4y^2 + 9z^2 \leq 1$.

8. Compute the volume of the region given by $x^2 + y^2 + 4z^2 \leq 1$.

4.6 Integration practice

Exercises.

1. Find the optimal strategy for computing the following integrals. If the strategy is not obvious, then set up the integrals in all of the ways you can: by means of Fubini's Theorem, by means of polar or cylindrical coordinates, by means of spherical coordinates, try a linear change of coordinates. Based on what comes out decide what the best of these methods is, and decide what that is that makes it the best method. **You are not expected to actually compute the integrals, but you are strongly encouraged to pursue the computations until you get to the point where you can freely say "I know how to finish this problem".**

   (a) The integral
   $$\iint_R (x+y)\, dA$$
   where R is the parallelogram spanned by the vectors $\langle 1,1\rangle$ and $\langle 0,2\rangle$ with the base point at $(2,0)$;

   (b) The integral
   $$\iint_R \frac{y}{x}\, dA$$
   where R is region between the parabola $y = x^2 - 4x + 3$ and the line $y = x - 1$;

   (c) The integral that computes the volume enclosed between $z = \sqrt{x^2+y^2}$ and $z = x^2+y^2$;

   (d) The average value of the function $f(x,y) = x^2+y^2$ over the upper unit half-ball centered at the origin;

   (e) $\displaystyle\iint_{\mathbb{R}^2} \frac{dA}{(1+x^2+y^2)^2}$;

   (f) $\displaystyle\iint_{\mathbb{R}^2} \frac{dA}{(1+x^2+4y^2)^2}$;

   (g) The integral that computes the volume inside $x^2+y^2+z^2 = R^2$ but above the plane $z = \frac{R}{2}$.

   (h) The integral
   $$\iint_R \frac{1}{\sqrt[3]{x^2+y^2}}\, dA$$
   where R is the portion of the unit disk located in the first quadrant;

   (i) The integral that computes the volume contained in the intersection of two unit balls: one centered at the origin, and one centered at $(0,0,1)$;

   (j) $\displaystyle\iiint_\Omega x\, dV$ where Ω is the unit ball centered at $(1,0,0)$;

   (k) The integral that computes the volume contained between the planes $z = \pm 1$ and inside $x^2+y^2-z^2 = 1$;

   (l) The integral that computes the volume contained between the planes $z = \pm 2$ and inside the hyperboloid $4x^2+y^2-z^2 = 4$;

(m) The integral that computes the volume of the donut formed when the circle of radius 2 centered at $(3,0)$ rotates around the y-axis;

(n) The integral $\iiint_\Omega (x+y+z)\,dV$ where Ω is the parallelotope spanned by the vectors $\langle 1,1,0\rangle$, $\langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$ whose initial point is at the origin;

(o) The integral $\iint_R x\,dA$ where R is the region within the first quadrant which is inside the ellipse $x^2+4y^2=1$;

(p) The integral $\iiint_\Omega (x^2+y^2+z^2)\,dV$ where Ω is the pyramid whose base is the diamond with vertices $(\pm1,0,0)$, $(0,\pm1,0)$, and whose tip is at $(0,0,1)$.

(q) The integral $\iint_{\mathbb{R}^2} e^{-(2x-1)^2-(4y+1)^2}\,dA$;

(∞) The integral $\iiint_{\mathbb{R}^3} e^{-x^2-y^2-z^2}\,dV$.

4.7 Integrating on parametrized surfaces

Key Ideas.

- *We have two ways to compute the induced area element for a surface in* $\mathbb{R}^3$*:*
  - *Use dot products:* $dA = \sqrt{\det \begin{pmatrix} \partial_u \cdot \partial_u & \partial_u \cdot \partial_v \\ \partial_v \cdot \partial_u & \partial_v \cdot \partial_v \end{pmatrix}}\, du\, dv$
  - *Use cross product:* $dA = \|\partial_u \times \partial_v\| du\, dv$
- *Classic examples:*
  - *Compute the surface area of a cone of height* h *and radius* r*.*
  - *Compute the surface area of a sphere of radius* r*.*
- *The city of Portland is at roughly* $45°$ *north. What percent of the earth's surface is further north than Portland?*

Exercises.

1. Construct a parametrization for the surface of the ellipsoid $x^2 + y^2 + 9z^2 = 1$. Then use integration to compute the surface area.
2. Compute the surface area of the parabolic bowl given by $x^2 + y^2 = z$ and $0 \le z \le 4$.
3. Compute the surface area of the torus having large radius 4 and small radius 1.
4. Compute the surface area of the helicoidal surface given by

$$\begin{aligned} x(u,v) &= u\cos(v) \\ y(u,v) &= u\sin(v) \\ z(u,v) &= v \end{aligned} \qquad \begin{aligned} 0 \le u \le 3 \\ 0 \le v \le 2\pi \end{aligned}$$

5. Compute $\iint_S z^2\, dA$ where S is the surface of the cylinder of radius 2 centered along the z-axis, contained between the planes $z = \pm 1$. The top and bottom of the cylinder are not included.
6. Compute $\iint_S (x^2 + y^2)\, dA$ where S is the surface of the sphere of radius 2 centered at the point $(2, 0, 0)$.
7. An earring is made out of a very thin material and is made in the shape of a helicoid with the parametric equations $x(u,v) = u\cos(v)$, $y(u,v) = u\sin(v)$, $z(u,v) = v$, where $0 \le u \le \frac{1}{4}$, $0 \le v \le 6\pi$. (Here u is measured in inches, while v is measured in radians.) The weight density of the material at the location with coordinates (u,v) is $\varrho = \frac{1}{4\sqrt{u^2+1}}$ ounces per square inch. How heavy is the earring?

4.8 Work and flux along paths

Key Ideas.

- *A curve C in the plane $\mathbb{R}^2$ is* ***oriented*** *if we specify which direction is "positively along" and which direction is "positively across". Mathematically, we do this with a* ***frame along C****, which is comprised of a consistent unit tangent vector $\hat{T}$ along C and a consistent unit normal vector $\hat{N}$ along C.*
- *The circle and the figure-eight curves are good examples.*
- *If C is an oriented curve, we write $-C$ for the same curve, but with opposite orientation.*
- *Suppose we have an oriented curve C in the plane and a vectorfield $\vec{V}$ on $\mathbb{R}^2$.*
  - *The* ***work integral*** *$\int_C \vec{V} \cdot \hat{T}\, ds$ represents the total extent to which $\vec{V}$ "pushes positively along the curve."*
  - *The* ***flux integral*** *$\int_C \vec{V} \cdot \hat{N}\, ds$ represents the total extent to which $\vec{V}$ "pushes positively across the path."*

  Note that the orientation determines the sign of the work and flux integrals.
- *If C is parametrized by $x(t)$ and $y(t)$ then*

$$\hat{T}\, ds = \pm \frac{\langle x'(t), y'(t)\rangle}{\|\langle x'(t), y'(t)\rangle\|} \|\langle x'(t), y'(t)\rangle\|\, dt = \pm\langle x'(t), y'(t)\rangle dt$$

  and

$$\hat{N}\, ds = \pm \frac{\langle y'(t), -x'(t)\rangle}{\|\langle y'(t), -x'(t)\rangle\|} \|\langle y'(t), -x'(t)\rangle\|\, dt = \pm\langle y'(t), -x'(t)\rangle dt$$

  The choice $\pm$ determines the orientation.

Exercises.

1. Let C denote the unit circle $x^2 + y^2 = 1$, oriented counterclockwise, and consider the vector field $\vec{V}(x, y) = \langle y, 0\rangle$.
   (a) Plot the circle and the vector field *by hand* on the same diagram.
   (b) Based on your sketch, give a written description of the extent to which the vector field $\vec{V}$ flows along C. (A couple of sentences is enough.)
   (c) Find an explicit formula for the unit tangent vector field $\hat{T}$ to C.
   (d) Find an explicit formula which describes, infinitesimally at each point, the extent to which the vector field $\vec{V}$ flows along C. Do make sure that this quantitative information matches with the qualitative description you provided previously.
   (e) What effect does replacing C with $-C$ has?
2. Let C denote the counter-clockwise unit circle centered at the origin. Furthermore, let

$$\vec{V}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

(a) Plot C and $\vec{V}$ by hand on the same diagram.

(b) Based on your sketch, give a prediction/estimate of $\int_C \vec{V} \cdot \hat{T}\, ds$. (A sentence or so explaining your logic will suffice.)

(c) Parametrize C and explicitly compute $\int_C \vec{V} \cdot \vec{T}\, ds$. Does your computation agree with your previous estimate?

3. Compute $\int_C \vec{V} \cdot \hat{T}\, ds$ where:

(a) C is the portion of the curve $y = \frac{1}{2}(e^x + e^{-x})$ between $x = -1$ and $x = 1$, oriented from left to right, and $\vec{V}(x, y) = \langle -y, 1\rangle$.

(b) C is the top half of the ellipse $4x^2 + y^2 = 1$, oriented counter-clockwise, while $\vec{V}(x, y) = \langle x + y, -x + y\rangle$.

(c) C is the line segment between $(1, 0, 1)$ and $(-1, -2, -1)$, while the vector field $\vec{V}$ is given by $\vec{V}(x, y, z) = \langle x + 2y, x + 2y, 1\rangle$.

4. Let C denote the unit circle $x^2 + y^2 = 1$ and let $\vec{V}(x, y) = \langle x^2, 0\rangle$.

(a) Plot the circle and the vector field on the same diagram. Hand-drawn sketch will suffice.

(b) Based on your sketch from the above, qualitatively assess (i.e. eyeball) the extent to which the vector field $\vec{V}$ flows outside of the region bounded C. Do so both on a local (e.g. infinitesimal) and global level.

(c) Find an explicit formula for the outward pointing unit normal vector field $\hat{N}$.

(d) Find an explicit formula which describes the infinitesimal out-flux of $\vec{V}$ across C. Do make sure that this quantitative information matches with the qualitative description you provided in part (b).

(e) What is the (total) out-flux of $\vec{V}$ across C? What is the (total) in-flux?

5. Let C denote the unit circle centered at the origin and let $\vec{V}(x, y) = \langle -x, -y\rangle$.

(a) Plot C and $\vec{V}$ on the same diagram. Hand-drawn sketch will suffice.

(b) Based on your sketch, try to qualitatively assess (i.e. eyeball) the outward flux $\int_C \vec{V} \cdot \hat{N}\, ds$.

(c) Parametrize C and explicitly compute $\int_C \vec{V} \cdot \hat{N}\, ds$. Do make sure that your computation lines up with the description from part (b).

6. Compute the out-flux and the in-flux of the vector field $\vec{V}(x, y) = \langle y, x\rangle$ across the unit circle centered at the origin.

4.9 Flux across surfaces

Key Ideas.

- *A surface S inside $\mathbb{R}^3$ is **oriented** if there exists a unit normal vector field $\hat{N}$ on S. The choice of $\hat{N}$ is called the **orientation**.*
- *Suppose we have an oriented surface S (with unit normal vector field $\hat{N}$) and a vector field $\vec{V}$. The **flux integral** $\iint_S \vec{V} \cdot \hat{N}\, dA$ represents the total extent to which $\vec{V}$ "pushes across the surface." The choice of orientation $\hat{N}$ determines which direction is considered positive.*
- *If S is parametrized by coordinates u and v, then $\hat{N}\, dA = \pm\partial_u \times \partial_v\, du\, dv$.*

Exercises.

1. Let S denote the unit sphere $x^2 + y^2 + z^2 = 1$ and let $\vec{V}(x, y, z) = \langle 1, 0, 0\rangle$.
   (a) Sketch by hand the sphere and the vector field on the same diagram.
   (b) Based on your sketch estimate the extent to which the vector field $\vec{V}$ flows outside of the unit ball. Do so both on a local (infinitesimal) and global level.
   (c) Find an explicit formula for $\hat{N}\, dA$.
   (d) Find an explicit formula which describes the infinitesimal out-flux (at each point) of $\vec{V}$ across S. Does your result match the your previous estimate?
   (e) What is the (total) out-flux of $\vec{V}$ across S? What is the (total) in-flux?
2. Compute the following:
   (a) Let S be the upper dome of the sphere of radius 2 centered at the origin. (The equatorial base is not included.) Find the outward flux of $\vec{V}(x, y, z) = \langle 0, y, z\rangle$ across S.
   (b) Let S be the surface of the cone $z = \sqrt{x^2 + y^2}$ located below the $z = 1$. (Note: the circular base / top of the cone is not included.) Find the inward flux of the vector field $\vec{V} = \langle x, y, 2z\rangle$ across the surface S.
   (c) Let S be the surface of the solid formed as an intersection of the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. Find the outward flux of the vector field $\vec{V} = \langle 0, 0, z\rangle$ across S. (Hint: break the flux integral into the sum of two.)