3.1 The Jacobi matrix and linearization

Key Ideas.

- *Recall Taylor's theorem*

$$f(t_* + \Delta t) = f(t_*) + f'(t_*)\Delta t + \textit{remainder}$$

The best linear approximation of f *near* t_* *is*

$$L(\Delta t) = f(t_*) + f'(t_*)\Delta t.$$

- *Multivariable version*

$$\begin{aligned} &T(u_* + \Delta u, v_* + \Delta v, \dots) \\ &= T(u_*, v_*, \dots) + \begin{pmatrix} | & | & \\ \frac{\partial T}{\partial u}(u_*, v_*, \dots) & \frac{\partial T}{\partial v}(u_*, v_*, \dots) & \cdots \\ | & | & \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \vdots \end{pmatrix} \\ &\qquad + \textit{remainder} \end{aligned}$$

Write as

$$T(u_*+\Delta u, v_*+\Delta v, \dots) = T(u_*, v_*, \dots) + DT(u_*, v_*, \dots) \begin{pmatrix} \Delta u \\ \Delta v \\ \vdots \end{pmatrix} + \textit{remainder}$$

- *The multivariable derivative is the* ***Jacobi matrix***

$$DT = \begin{pmatrix} | & | & \\ \frac{\partial T}{\partial u} & \frac{\partial T}{\partial v} & \cdots \\ | & | & \end{pmatrix} = \begin{pmatrix} | & | & \\ \partial_u & \partial_v & \cdots \\ | & | & \end{pmatrix}$$

If we write $T(u, v, \dots) = (x(u, v, \dots), y(u, v, \dots), \dots)$ *then*

$$DT = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \cdots \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

- *The best linear approximation of* T *at the point* (u_*, v_*) *is given by the affine transformation*

$$L(\Delta u, \Delta v) = T(u_*, v_*) + DT(u_*, v_*) \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

Exercises.

1. Consider the transformation
$$T(u,v) = (u\cos(v), u\sin(v)).$$
   (a) Draw, as best you can, the transformation T.
   (b) Compute the Jacobi matrix for T.
   (c) Use the Jacobi matrix to find the best linear approximation of T at the point $(u,v) = (1, \pi/3)$.
   (d) Draw, as best you can, the linear approximation. Be sure to indicate the coordinate vector fields in your picture.
2. Consider the transformation
$$T(u,v) = (u\cos(v), u\sin(v), u^2).$$
   (a) Draw, as best you can, the transformation T.
   (b) Compute the Jacobi matrix for T.
   (c) Use the Jacobi matrix to find the best linear approximation of T at the point $(u,v) = (1, \pi/4)$.
   (d) Draw, as best you can, the linear approximation. Be sure to indicate the coordinate vector fields in your picture.
3. Consider the transformation
$$T(u,v) = (\cos(v), \sin(v), u).$$
   (a) Draw, as best you can, the transformation T.
   (b) Compute the Jacobi matrix for T.
   (c) Use the Jacobi matrix to find the best linear approximation of T at the point $(u,v) = (1, \pi/4)$.
   (d) Draw, as best you can, the linear approximation. Be sure to indicate the coordinate vector fields in your picture.
4. Consider the transformation
$$T(t) = (\cos(t), \sin(t)).$$
   (a) Draw, as best you can, the transformation T.
   (b) Compute the Jacobi matrix for T.
   (c) Use the Jacobi matrix to find the best linear approximation of T at the point $t = \pi/4$.
   (d) Draw, as best you can, the linear approximation. Be sure to indicate the coordinate vector fields in your picture.

5. Consider the transformation

$$T(t) = (\cos(t), \sin(t)).$$

   (a) Draw, as best you can, the transformation T.
   (b) Compute the Jacobi matrix for T.
   (c) Use the Jacobi matrix to find the best linear approximation of T at the point $t = \pi/4$.
   (d) Draw, as best you can, the linear approximation. Be sure to indicate the coordinate vector fields in your picture.

6. Consider the transformation

$$T(x, y) = x^2 - y^2.$$

   (a) Draw, as best you can, the transformation T.
   (b) Compute the Jacobi matrix for T.
   (c) Use the Jacobi matrix to find the best linear approximation of T at the point $(x, y) = (2, 3)$.
   (d) Draw, as best you can, the linear approximation. Be sure to indicate the coordinate vector fields in your picture.

7. Consider the transformation

$$T(u, v) = (u^2 - v^2, 2uv),$$

   where $u \geq 0$.

   (a) Draw, as best you can, the transformation T.
   (b) Compute the Jacobi matrix for T.
   (c) Use the Jacobi matrix to find the best linear approximation of T at the point $(u, v) = (1, 1)$.
   (d) Draw, as best you can, the linear approximation. Be sure to indicate the coordinate vector fields in your picture.