Chapter 1

First order equations

1.1 Introduction: What is a differential equation?

We introduce the question "what is a differential equation?" from the perspective of population modeling.

Example 1.1.1. *Suppose that a population of bacteria is increasing at a constant rate of* 25 *million per year. This means that the ratio of the change in population to the change in time is* 25*. Let* P *represent the size of the population and let* t *represent time. Since the population changes in time,* P *is a function of* t*. Our assumption that the bacteria population is increasing at a constant rate in time can be phrased as*

$$\frac{dP}{dt} = 25 \quad \textit{or} \quad P'(t) = 25. \tag{1.1.1}$$

(Note that we suppress the units. However, the units are important!)

The equation (1.1.1) is an example of an ***ordinary differential equation (ODE)***. Let's unpack this fancy title:

- (1.1.1) is an *equation* in that is states that two mathematical objects are equal. In this case, the equation states that the function $P'(t)$ and the constant function 25 are the same. It is important that the equality in (1.1.1) is understood in the sense of *functions*.
- (1.1.1) is a "differential" equation because the equation involves the derivative of the function P.
- the word "ordinary" indicates that the function P depends only on one variable (namely, t) and thus (1.1.1) involves ordinary derivatives (as opposed to partial derivatives).

We use differential equations to mathematically describe assumptions about rates of change.

Example 1.1.2. *Suppose that the rate of change of a population is oscillating periodically between* 2 *and* 3*. If we assume that the oscillation is sinusoidal with period* 2π*, then we can describe this situation with the differential equation*

$$\frac{dP}{dt} = \frac{5}{2} + \frac{1}{2}\sin(t).$$

How would you change this equation to adjust the period of oscillation?

In the previous examples, the rate of change of the population is directly specified. However, in many real-world situations, the rate of change of a population at any given moment depends on the size of the population at that moment.

Example 1.1.3. *Suppose that the rate of change of a population is three precent of the population per year. Mathematically, we can describe this situation using the equation*

$$\frac{dP}{dt} = 0.03P.$$

What are the units on 0.03*?*

The differential equation in the previous examples illustrates a sort of "feedback cycle" that occurs when the rate of change of a population depends on the population: the larger the population, the faster the growth rate... which makes the population even larger, and thus grow faster and faster! Mathematically, equations with this type of feedback cycle are much more interesting than the first two equations we saw – you can't just integrate the growth rate function to find the population! (Why not?)

In this class we approach the study of differential equations as follows. We begin by developing basic theory and techniques, while focusing on ODEs that model a single population. For equations of special forms, we are able to find exact formulas for solutions to equations. For other equations, we use qualitative methods to study solutions, and use numerical methods to find approximate solutions. We conclude this introductory part of the course with a discussion of the "Fundamental Theorem" of ODEs. The remainder of the course is used for in-depth study of three topics: systems of interacting populations, conservation of energy, and oscillations.

1.2 Constructing differential equations

Basic growth model

We constructed the differential equation (1.1.1) by assuming that the derivative of the function $P(t)$ was constant. The derivative $P'(t)$ represents the ***absolute growth rate***, also called the ***absolute rate of change***, of the function P. Rates of change, however, are often discussed in terms of percentages rather than in absolute terms. Such percentages refer to the ***relative growth rate***, also called the ***percent rate of change***, which is defined by

$$\text{relative growth rate of } P = \frac{P'(t)}{P(t)} = \frac{1}{P}\frac{dP}{dt}.$$

Notice that the units of the relative growth rate are simply 1 divided by the units of t; the units of P cancel out.

We can construct differential equations by making assumptions about either the absolute or relative growth rate of a population. The simplest assumption, which is the one made in the previous section, is that the absolute growth rate is constant. The corresponding differential equation is

$$\frac{dP}{dt} = r, \tag{1.2.1}$$

where r is some constant. The differential equation (1.2.1) is called the ***constant growth model*** because it describes a population with a constant absolute growth rate.

Alternatively, we may assume that the relative growth rate is constant. The resulting differential equation is

$$\frac{1}{P}\frac{dP}{dt} = r, \tag{1.2.2}$$

where r is some constant. The differential equation (1.2.2) is called the ***basic growth model***. The basic growth model describes situations where population P is growing at r percent per unit of time.

Notice that the constant r in (1.2.1) has different units from the constant r appearing in (1.2.2)!

Example 1.2.1. *Suppose a fixed amount of money is placed in to a bank account that earns* 3% *annual interest. Let* $A(t)$ *be the value of the account at time* t*, where we measure* t *in years. Then* A *must satisfy*

$$\frac{1}{A}\frac{dA}{dt} = 0.03.$$

Notice that the amount of money originally placed in to the account does not affect this differential equation. The equation only describes how the amount is changing, not what the amount actually is!

Multiplying both sides of (1.2.2) by P, we can re-write the basic growth model as

$$\frac{dP}{dt} = rP. \tag{1.2.3}$$

We can interpret the equation (1.2.3) as the statement that the absolute rate of change of P is proportional to P. (Remember that "is proportional to" means "is equal to a constant times.") Thus there are two different assumptions that both lead to basic growth model equation.

It is useful, and interesting, to consider variations on the basic growth model that incorporate additional features.

Example 1.2.2. *Suppose that the bacteria population in my jar of yogurt has a relative growth rate of* 5% *per hour. Additionally, I constantly remove yogurt from the jar in such a way that bacteria are being removed at a rate of* 7 *million per hour.*

We can construct differential equation modeling this situation as follows. Let $P(t)$ *be the population of bacteria in the jar at time* t*, measured in millions; let the time* t *be measured in hours. We assume that the absolute rate of change of* P *is the sum of two terms, the first coming from the relative growth rate and the second coming from the removal of the bacteria. The resulting differential equation is*

$$\frac{dP}{dt} = 0.05P - 7. \tag{1.2.4}$$

The term $0.05P$ *comes from the assumption on the relative growth rate of the bacteria, while the term* -7 *comes from the removal of the bacteria. Notice that each of the two terms on the right side of* (1.2.4) *have units of millions per hour, which is the same as the units of the left side. It is always important that terms being added together have the same units!*

More generally, if population modeled by P grows with relative growth rate r and is subject to "migration" determined by the function f, then the P satisfies

$$\frac{dP}{dt} = rP + f. \tag{1.2.5}$$

The function f is sometimes called the ***forcing term***, and thus we can call (1.2.5) the ***forced basic growth model***.

Example 1.2.3. *What differential equation describes the assumption that the relative growth rate is proportional to the population size?*

This assumption can be expressed as

$$\frac{1}{P}\frac{dP}{dt} = rP,$$

where r is the constant of proportionality and has units of

$$\frac{1}{(\textit{units of } P)(\textit{units of } t)}.$$

We can rewrite the equation as

$$\frac{dP}{dt} = rP^2.$$

Logistic model

Another important variation on the basic growth model comes from making the assumption that the population being studied lives in a habitat that can only sustain a finite size population, and that the relative growth rate of the population is proportional to the percent of available habitat. In order to construct a differential equation from this assumption, let $P(t)$ describe the size of the population and let K be the population size that the habitat can sustain. The percent of habitat that is available is given by the function

$$1 - \frac{P}{K}.$$

Thus the assumption that the relative growth rate is proportional to the percent of available habitat leads to the differential equation

$$\frac{1}{P}\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right),$$

where r is some constant. We can rewrite this equation as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{1.2.6}$$

The equation (1.2.6) is called the ***logistic growth model***.

Notice that the constant r in the logistic growth model (1.2.6) has the same units as the constant r in the basic growth model (1.2.2). In fact, if P is very small relative to K, then the term P/K in (1.2.6) is very close to zero.

Thus when the habitat can support a very large population size (relative to the current population size), then the logistic growth model is approximated by the basic growth model. (We make this idea of one differential equation approximating another a bit more precise in the next chapter.) For this reason we say that the constant r in the logistic model is the ***ideal relative growth rate***, or ***relative growth rate under ideal conditions***.

Example 1.2.4. *Suppose that the milk in my yogurt jar can sustain a population of* 9 *million bacteria, and that the bacteria has ideal relative growth rate of* 15% *per hour. Then we can model the population of bacteria in the jar by the differential equation*

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{9}\right),$$

where we measure P *in millions and* t *in hours.*

Notice that in the previous example I chose to measure the population P in millions. We could have decided to measure the population in units of individual bacteria, in which case the differential equation would be

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{9000000}\right).$$

Both of these differential equations describe the same physical situation, but the first one is much easier to work with since the numbers are of more "reasonable size." In general, it is advisable to choose units so that the numbers appearing in the resulting differential equations are as "reasonable" as possible.

It is possible to modify the logistic model in ways analogous to the modification of the basic growth model appearing in Example 1.2.2.

Activity 1.2.1. *Construct a differential equation describing the following situation: A population has ideal relative growth rate of* 7% *per year and lives in a habitat that can sustain a population of* $12,000$ *individuals. Furthermore, individuals are moving in to the habitat at a constant rate of* $3,000$ *per year.*

Activity 1.2.2. *Construct a differential equation describing the following situation: A population has ideal relative growth rate of* 3% *per month and lives in a habitat that can sustain a population of* 500 *individuals. Furthermore, individuals are continuously leaving the habitat at a rate of* 5% *per month.*

Initial conditions

In some (though not all) modeling scenarios we not only have assumptions about how the population is changing, we also have information about the initial size of the population. Mathematically, this information tells us the value for $P(t_0)$, where t_0 is the initial time. This information is called an ***initial condition***.

Example 1.2.5. *Suppose Paul puts 5 grams of dry yeast in to his bread dough. Dry yeast has roughly 10 billion cells per gram, so the initial size of the population is 50 billion cells. During the first part of the fermenting process, the basic growth model can be used to describe the population of yeast. Assuming a relative growth rate of* $1/2\%$ *per minute, obtain a model that has both an ODE and a initial condition:*

$$\frac{dP}{dt} = 0.005P \qquad P(0) = 50,$$

where time t *is measured in minutes and the population* P *is measured in billions of cells.*

Mixing problems

All of the differential equations constructed above are population models. Differential equations can be used to describe a wide variety of phenomena beyond population dynamics. Later in the course we address a number of equations arising in physics, while in the first part of the course we focus primarily on population models. The concept of a "population," however, can encompass a wide range of phenomena, as the following examples demonstrate.

Example 1.2.6. *Consider a tank with capacity* 100 *gallons, containing a mixture of fresh water and salt. The contents of the tank are being drained at a rate of* 5 *gallons per minute. Simultaneously, salt water having concentration of* 8 *ounces per gallon is being pumped in to the tank. We assume that the mixture in the tank is being stirred so that the salt concentration in the tank is uniform.*

We can model this situation by considering the amount of salt in the tank to be a "population." Let $S(t)$ *be the amount of salt (measured in ounces) in the tank at time* t *(measured in minutes). Then the rate of change of salt in the tank satisfies*

$$\frac{dS}{dt} = (\textit{rate of salt in}) - (\textit{rate of salt out}).$$

We can easily find expressions for the rate of salt entering and leaving the tank as follows:

$$\begin{aligned}(\textit{rate of salt in}) &= (\textit{rate of liquid entering tank})\\ &\qquad \times (\textit{concentration of salt in the incoming liquid})\\ &= \left(\frac{5\ \textit{gallons}}{\textit{minute}}\right)\left(\frac{8\ \textit{ounces}}{\textit{gallon}}\right)\\ &= 40\,\frac{\textit{ounces}}{\textit{minute}}\end{aligned}$$

Similarly, we compute that

$$\begin{aligned}(\textit{rate of salt out}) &= (\textit{rate of liquid leaving tank})\\ &\qquad \times (\textit{concentration of salt in the tank})\\ &= \left(\frac{5\ \textit{gallons}}{\textit{minute}}\right)\left(\frac{S\ \textit{ounces}}{100\ \textit{gallons}}\right)\\ &= \frac{1}{20}\,S\,\frac{\textit{ounces}}{\textit{minute}}\end{aligned}$$

Thus the differential equation that describes the change in S is

$$\frac{dS}{dt} = 40 - \frac{1}{20}S. \tag{1.2.7}$$

Notice that (1.2.7) takes the form of the forced basic growth model (1.2.5) with relative growth rate $r = -1/20$ and forcing $f = 40$.

Activity 1.2.3. *Paul's brother purchases 22 ounce chocolate drink and begins drinking at a rate of 3 ounces per minute. Paul begins to pour strawberry drink, which contains 10% fruit, in to his brother's glass at a rate of 1 ounce per minute. Find a differential equation that models the amount of fruit in the brother's glass as a function of time.*

Exercises

Exercise 1.2.1. *Construct a differential equation which models the following situations. Include an initial condition if appropriate.*

1. *An investment $y(t)$ grows with relative growth rate of 5% per year;*
2. $1000 *is invested with annual interest rate of 5%;*
3. *Continuous deposits are made into an account at the rate of* $1000 *a year. In addition to these deposits, the account earns 7% interest per year.*

4. *Paul takes out a loan with an annual interest rate of* 6%. *Continuous repayments are made totaling* \$1000 *per year.*

5. *A fish population under ideal conditions grows at a relative growth rate* k *per year. The carrying capacity of their habitat is* N *and* H *fish are harvested each month.*

6. *A fish population under ideal conditions grows at growth rate* k *per year. The carrying capacity of their habitat is* N *and one quarter of the fish population is harvested annually.*

7. *Due to the pollution problems the relative growth rate of a fish population is a decreasing exponential function of time.*

Exercise 1.2.2. *Some quantity* f *changes with time, is measured in gallons, and is modeled by the differential equation*

$$\frac{df}{dt} = kf\left(1 - \frac{f}{M}\right),$$

where the time variable t *is measured in months and the parameter* M *is also measured in gallons.*

1. *In what units is* $\frac{df}{dt}$ *measured?*

2. *In what units is* $1 - \frac{f}{M}$ *measured?*

3. *In what units is* $f\left(1 - \frac{f}{M}\right)$ *measured?*

4. *In what units is the parameter* k *measured?*

Exercise 1.2.3. *A variable quantity* $r = r(t)$ *is measured in grams per liter and is modeled by the differential equation*

$$\frac{dr}{dt} = -\frac{kr}{c + r}.$$

The time variable t *is measured in seconds. Figure out the units for the parameters* k *and* c.

Exercise 1.2.4. *We consider a large urn of coffee in cafeteria. A student is draining coffee in to her thermos while at the same time the brewing machine is adding fresh coffee. We want to keep track of the amount of caffeine (in milligrams) in the urn during this process.*

Suppose the following:

- *There is initially* 4 *liters of coffee in the urn, with a caffeine concentration of* 100 *mg per liter.*
- *Starting at time* $t = 0$ *the brewing machine is adding extra strong coffee, which has a concentration of* 250 *mg per liter. This new coffee is being added at a rate of* 25 *mL per minute.*
- *Starting at time* $t = 0$ *the student begins draining coffee out of the urn at a rate of* 30 *mL per minute.*

Write down a differential equation which models the amount of caffeine in the urn as a function of time.

Exercise 1.2.5. *In this problem we model the number of junk emails in Iva's inbox with a continuous function* $J(t)$*. Suppose the following:*

- *When the semester started* ($t = 0$)*, Iva had* 6000 *emails in her inbox;* 4000 *of them were junk.*
- *Email is continuously flowing in to Iva's email inbox at a rate of* 50 *per day;* 30% *of the emails are junk.*
- *Each day, Iva randomly picks* 20 *emails to deal with*[1] *– once they have been dealt with, she moves them out of her inbox.*

Write down a differential equation describing the number of junk emails in Iva's inbox.

Exercise 1.2.6. *A big mixing vat contains* 50 *liters of a mixture in which the concentration of a certain chemical is* 1.25 *grams per liter. This mixture is being diluted by another mixture in which the concentration of the same chemical is* 0.25 *grams per liter. Each minute* 6 *liters of the less concentrated mixture are poured into the vat and* 4 *liters of the resulting new mixture are drained out. Construct a differential equation modeling this process.*

[1] This is not actually true, of course.

1.3 What is a solution to a differential equation?

Activity 1.3.1. *Find all solutions to the system of equations*

$$x^2 + xy + x = 0,$$
$$-y^2 + xy + 2y = 0.$$

Be systematic... there should be four solutions!

The previous activity involves finding solutions to an *algebraic* equation. It's clear what we mean by a solution: the pair $(-1, 0)$ is a solution because when we plug in -2 for x and 0 for y, the system of equations is satisfied. In other words, a solution is something that "if we plug it in, it works."

The same principle of "plug it in and it works" applies to differential equations as well.

Example 1.3.1. *Consider the differential equation*

$$\frac{dy}{dt} = 5y. \tag{1.3.1}$$

Here the unknown is the function $y(t)$.

It is easy to verify that the function $y(t) = 8e^{5t}$ is a solution to (1.3.1). If we plug this function in to the left side, we get

$$\frac{d}{dt}\left[8e^{5t}\right] = 40e^t,$$

while if we plug it in to the right side we get

$$5\left[8e^{5t}\right] = 40e^t.$$

Since these two are the same, the function $y(t) = 8e^{5t}$ is a solution to (1.3.1).

Notice that there are in fact an infinite number of solutions to (1.3.1). The function $y(t) = Ce^{5t}$ is a solution for any value of C. Here C is what we call a free parameter. The solution $y(t) = Ce^{5t}$ that includes the free parameter C is called a general solution. We explain this terminology in more detail below.

Activity 1.3.2. *Consider the differential equation*

$$\frac{dy}{dt} = 6y + 3t.$$

Show that the function

$$y(t) = e^{6t} - \frac{1}{12} - \frac{1}{2}t.$$

is a solution.

Unfortunately, the principle of "plug it in and it works" does not work for all equations – this is the case both for algebraic equations and for differential equations.

Example 1.3.2. *Consider the algebraic equation*

$$e^x + x = 0. \tag{1.3.2}$$

No matter how we try, we cannot "isolate x" and find an closed-form expression for a solution to this equation.

However, it is straightforward to deduce that (1.3.2) does indeed have a solution. To see this, consider the function $f(x) = e^x + x$. It is easy to see that

$$f(-1) = e^{-1} - 1 < 0 \quad \text{and} \quad f(0) = e^0 = 1 > 0.$$

Thus by the intermediate value theorem, there must exist a number x_* between -1 and 0 where $f(x_*) = 0$. Clearly the number x_* is a solution to (1.3.2).

Notice that not only have we deduced indirectly that (1.3.2) has a solution, we also have shown one property of the solution, that the solution is between -1 and 0. We can easily deduce another property: since $f'(x) = e^x + 1 > 0$ we see that the function f is strictly increasing and thus there can be only one solution to (1.3.2).

Finally, we can use technology to obtain a numerical approximation of the solution.

The previous examples illustrates that for algebraic equations we have the following situation.

- Sometimes we are able to find an explicit formula for the solution.
- Even in situations where we cannot find an explicit formula for a solution, if we are clever we can often deduce whether or not solutions exist.
- In the case that there are solutions, we can also try to deduce how many solutions there are, as well as some properties of those solutions – all without having an explicit formula.
- Finally, in all cases we can use technology to obtain numerical approximations of solutions (when they exist).

The situation for differential equations is analogous to the situation for algebraic equations:

- Sometimes we will be able to find exact expressions for solutions to differential equations.
- When we cannot find exact solutions, we can still try to deduce indirectly whether solutions exist or not.
- In cases where we deduce that solutions exist, we can find clever ways to study those solutions.
- And we can use technology to obtain numerical approximation of solutions.

In the next several sections of this course we develop methods for studying differential equations that touches on each of these ideas.

Three useful examples

We now consider three differential equations for which we are able to find relatively simple formulas for solutions. They are

$$\frac{dy}{dt} = r, \tag{1.3.3a}$$

$$\frac{dy}{dt} = ry, \tag{1.3.3b}$$

$$\frac{dy}{dt} = ry^2; \tag{1.3.3c}$$

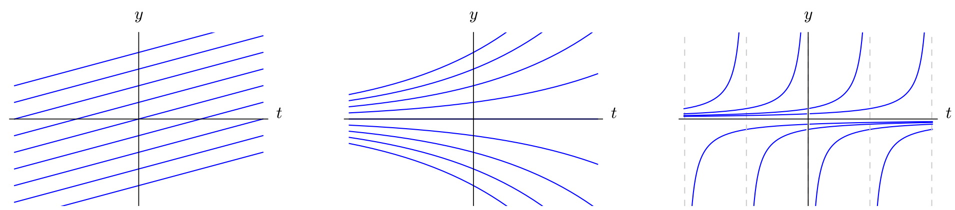
in each case r is a numerical constant.

Activity 1.3.3.

1. *Show that the function*
$$y(t) = C + rt$$
is a solution to (1.3.3a) *for any value of the constant* C*.*

2. *Show that the function*
$$y(t) = Ce^{rt}$$
is a solution to (1.3.3b) *for any value of the constant* C*.*

3. *Show that the function*
$$y(t) = \frac{1}{C - rt}.$$
is a solution to (1.3.3c) *for any value of the constant* C*.*



(a) Plots of $y(t) = C + rt$. (b) Plots of $y(t) = Ce^{rt}$. (c) Plots of $y(t) = 1/(C - rt)$.

Figure 1.3.1: Plots of the general solutions to (1.3.3a), (1.3.3b), (1.3.3c) for various values of the constant C; these plots assume that $r > 0$.

You are strongly encouraged to memorize the solutions presented in the this activity.

We can plot the solutions of the differential equations (1.3.3) for various values of C; see Figure 1.3.1. We notice three things from examining these plots.

First, the solution to a differential equation may not be defined for all values of the time parameter t – there is the possibility, illustrated by the plots of solutions to (1.3.3c), that the solution will ***blow up*** in finite time t_*, by which we mean $y(t) \to \pm\infty$ as $t \to t_*$. Thus the theory that we develop for differential equations must have two separate aspects: a local-in-time theory indicating that solutions exist and a separate theory regarding what happens as time progresses.

The second feature is that if we fix some time t_0 and fix some value y_0 we can always find a solution such that $y(t_0) = y_0$. Graphically, we see this by observing that there is a solution passing through the point (t_0, y_0) on the plot. Algebraically, we see that we can solve for an appropriate value of the constant C.

The condition $y(t_0) = y_0$ is called a ***initial condition***, meaning that we specify the initial value of the unknown function y. It is most common to specify the value at time $t_0 = 0$, but other initial times are also possible.

A solution to a differential equation with a free parameter (such as the constants C appearing above) is called the ***general solution*** to the differential equation if we can choose a value for the constant C to match any initial condition.

Activity 1.3.4. *For each of the general solutions in Figure 1.3.1 find the value of C so that the initial condition $y(0) = 3$ is satisfied.*

Finally, our third observation about Figure 1.3.1 is that solutions do not cross in the ty plane, meaning that for each initial condition (t_0, y_0) there exists only one solution that ever achieves that state. We interpret this to mean that there is a unique solution to the problem "Find a solution to the ODE satisfying a specific initial condition." The combination of an ODE and an initial condition is called a ***initial value problem***.

The general theory we develop below shows that these three points hold for most "reasonable" ODEs.

Exercises

Exercise 1.3.1.

1. *Determine if the function $y(t) = 1 + 2e^t$ is a solution of the differential equation*
$$\frac{dy}{dt} = (y-1)(y-2e^t).$$
2. *Determine if the function $y(t) = e^t - 2t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.*
3. *Determine if the function $y(t) = 2e^t - t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.*
4. *Determine for which values of the constant C the function $y(t) = Ce^t - t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.*

Exercise 1.3.2. *Verify that the function $y(t) = \sqrt{\frac{4-t^3}{3t}}$ solves the differential equation*
$$\frac{dy}{dt} = -\frac{t^2 + y^2}{2ty}.$$

Exercise 1.3.3. *Verify that for any constant C, the function*
$$y(t) = Ce^{6t} - \frac{1}{12} - \frac{1}{2}t$$
satisfies the differential equation
$$\frac{dy}{dt} = 6y + 3t.$$
Use this to find a function y that satisfies both
$$\frac{dy}{dt} = 6y + 3t \quad \text{and} \quad y(0) = 10.$$

Exercise 1.3.4. *Consider the differential equation*

$$\frac{dy}{dt} = \sqrt{y}.$$

1. *Show that for any constant C, the function*

$$y(t) = \left(\frac{1}{2}t + C\right)^2$$

   is a solution (at least when $t \geq -2C$).

2. *Show that the function $y(t) = 0$ is also a solution.*

3. *Find two different solutions to the initial value problem*

$$\frac{dy}{dt} = \sqrt{y} \qquad y(0) = 0.$$

Exercise 1.3.5. *In this problem we consider a differential equation that involves both first and second derivatives of the unknown function, namely*

$$t^2\frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y = 0.$$

Find those values of α such that the function $y(t) = t^\alpha$ solves the differential equation.

Exercise 1.3.6. *Consider the IVP*

$$\frac{dy}{dt} = y^3, \qquad y(0) = 1.$$

1. *Verify that the function*

$$y(t) = \sqrt{\frac{1}{1-2t}}$$

   solves this IVP.

2. *Graph this solution.*

3. *State the domain on which this solution is defined.*

4. *Describe what happens to the solution as the independent variable approaches the endpoints of the domain. Why can't the solution be extended for more time?*

Exercise 1.3.7.

1. *Find the solution to the initial value problem*

$$\frac{dy}{dt} = 1 - t^2, \qquad y(0) = 5.$$

2. *Find the solution to the initial value problem*

$$\frac{dy}{dt} = \cos t - \sin t, \qquad y(0) = \pi.$$

3. *Suppose that $f(t)$ is some function and y_0 is a constant. Explain why the solution $y(t)$ to the initial value problem*

$$\frac{dy}{dt} = f(t), \qquad y(0) = y_0$$

is given by

$$y(t) = y_0 + \int_0^t f(\tau)\, d\tau.$$

What's going on with the letter τ here?

1.4 Separable equations

In the previous section we mentioned that it is not always possible to find explicit formulas for solutions to ODEs. However, it is helpful to begin our exploration by studying a type of ODE that *does* admit explicit (or at least implicit) formulas for solution.

We say that an ODE is ***separable*** if it takes the special form of a product of "stuff involving y" times "stuff involving t". Schematically, we write this as

$$\frac{dy}{dt} = (\text{stuff involving } y)(\text{stuff involving } t). \tag{1.4.1}$$

Example 1.4.1. *The differential equation*

$$\frac{dy}{dt} = e^{3y}\cos(t)$$

is separable. The equation takes the form (1.4.1) *with*

$$\begin{aligned} \textit{stuff involving } y &= e^{3y} \\ \textit{stuff involving } t &= \cos(t). \end{aligned}$$

Let's now see why this is useful. We can re-write the ODE from the previous example as

$$\frac{1}{e^{3y(t)}}y'(t) = \cos(t).$$

We would like to integrate this equation from some initial time to some final time. In order to make our notation work out, we rewrite the ODE with a new letter for the time variable:

$$\frac{1}{e^{3y(\tau)}}y'(\tau) = \cos(\tau).$$

Now we integrate from time $\tau = t_0$ to time $\tau = t$, obtaining

$$\int_{t_0}^{t} \frac{1}{e^{3y(\tau)}}y'(\tau)d\tau = \int_{t_0}^{t} \cos(\tau)d\tau.$$

The integral on the right we can easily evaluate. For the integral on the left, it is helpful to make a change of variables. Using the "u-substitution" $u = y(t)$ we have $du = y'(t)dt$. Thus our equation becomes

$$\int_{y(t_0)}^{y(t)} \frac{1}{e^{3u}}du = \int_{t_0}^{t} \cos(\tau)d\tau.$$

Using the Fundamental Theorem of Calculus yields

$$-\frac{1}{3}e^{-3u}\Big|_{y(t_0)}^{y(t)} = \sin(\tau)\Big|_{t_0}^{t}.$$

This can be rewritten as

$$e^{-3y(t)} = e^{-3y(t_0)} - 3\sin(t) + 3\sin(t_0),$$

from which we can solve for $y(t)$*:*

$$y(t) = -\frac{1}{3}\ln\left(e^{-3y(t_0)} - 3\sin(t) + 3\sin(t_0)\right)$$

It is straightforward to see how to generalize the procedure in the previous example to other separable equations. Furthermore, if we have an initial value problem that gives us explicit values of t_0 and y_0, then the procedure automatically gives us the solution corresponding to those initial conditions.

There are two small modifications to the procedure that are commonly used. The first modification is that it is common to use the letter y, rather than the letter u, when doing the change of variable on the left side.

Activity 1.4.1. *Find the formula for the solutions to the following IVP:*

$$\frac{dy}{dt} = (1+y^2)(1+t^2), \qquad y(0) = 5.$$

The second modification is used when no initial conditions are present. Note that both the left and right side integrals contain a constant term, arising from evaluating the left side at $y(t_0)$ and evaluating the right side at t_0. If we don't have explicit values for t_0 and $y(t_0)$, then we can just use the letter C to stand in for the constants. Later, if we do happen to have initial conditions, we can solve for the constant.

In this modification, it is common to ignore the bounds of integration and to recycle the letter C. This requires us to be a bit careful, as the following example illustrates.

Example 1.4.2. *Consider the IVP*

$$\frac{dy}{dt} = 6ty^2, \qquad y(0) = 4.$$

First, let's ignore the initial condition and focus on the ODE

$$\frac{dy}{dt} = 6ty,$$

which we rewrite as

$$\frac{1}{y(t)}y'(t) = 6t.$$

Integrating, but ignoring the limits of integration, we obtain

$$\int \frac{1}{y(t)}y'(t)dt = \int 6tdt.$$

Changing variables on the left side yields

$$\int \frac{1}{y}dy = \int 6tdt.$$

Finding anti-derivatives, we obtain

$$\ln(y(t)) = 3t^2 + C. \tag{1.4.2}$$

Exponentiating both sides, we find that the solution is

$$y(t) = e^C e^{3t^2}.$$

Since e^C is a constant, we simply write the solution as

$$y(t) = Ce^{3t^2}. \tag{1.4.3}$$

Note that here we are using C to represent a different constant as before. The expression (1.4.3) is the general solution to the ODE.

Now we use (1.4.3) to address the initial value problem. We know that we want $y(0) = 4$. Thus we must have

$$4 = Ce^{3(0)^2},$$

from which we deduce that $C = 4$. Thus the solution to the IVP is

$$y(t) = 4e^{3t^2}.$$

Activity 1.4.2. *Find a formula for the general solution to*

$$\frac{dy}{dt} = (1 - y^2)(1 + t^2).$$

Activity 1.4.3. *Find a formula for the general solution to*

$$\frac{dy}{dt} = \frac{ty}{1 + t^2}.$$

The logistic population model

An important example of a separable equation is the logistic population model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \tag{1.4.4}$$

where both r and K are positive. When analyzing with this equation, it is helpful to work using the dimensionless variable

$$y(t) = \frac{P(t)}{K},$$

which measures the population $P(t)$ as a percent of the carrying capacity K. Dividing (1.4.4) by K, we replace each instance of P/K by y in order to obtain

$$\frac{dy}{dt} = ry(1-y). \tag{1.4.5}$$

The integral for this equation is

$$\int \frac{1}{y(1-y)}\,dy = \int r\,dt. \tag{1.4.6}$$

In preparation for integration, we use partial fractions to write

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Thus the integral becomes

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int r\,dt.$$

Integrating we find

$$\ln\left(\frac{y}{1-y}\right) = rt + C.$$

We apply the exponential to both sides and replace e^C by C, obtaining

$$\frac{y}{1-y} = Ce^{rt}. \tag{1.4.7}$$

Solving algebraically for y we find the following formula for y:

$$y(t) = \frac{Ce^{rt}}{1+Ce^{rt}}. \tag{1.4.8}$$

In terms of the variable P, this corresponds to

$$P(t) = \frac{Ce^{rt}}{1 + Ce^{rt}} K.$$

We can interpret the constant C by evaluating (1.4.7) at $t = 0$ to find

$$C = \frac{y_0}{1 - y_0};$$

here

$$y_0 = y(0) = \frac{P(0)}{K}$$

is the percent of the habitat that is initially full and

$$1 - y_0 = 1 - \frac{P(0)}{K}$$

is the precent of the habitat that is initially empty. Thus the constant C represents the ratio of these two percentages. We consider two "generic" cases and three special cases.

The first generic case is when $0 < P(0) < K$, which we interpret at the statement that initially the habitat has a population that is less than the carrying capacity. We call this the "underpopulated" case. In terms of y, this corresponds to $0 < y_0 < 1$, from which we deduce that the constant C is positive. We can compute from (1.4.8) that $y(t)$ exists for all $t \geq 0$ and that

$$\lim_{t\to\infty} y(t) = 1.$$

This corresponds to

$$\lim_{t\to\infty} P(t) = K,$$

which we interpret as the statement that in the long run, the population increases to fill the habitat. Note that mathematically we never have $P(t) = K$ (why not?). However, the difference between $P(t)$ and K eventually becomes less than one individual, in which case the two are "equal for all practical purposes."

Activity 1.4.4. *The second generic situation we consider is when $P(0) > K$, which we interpret as the statement that initially the population exceeds that which the habitat can sustainably support. What are possible values of C in this case? [Hint: It might be helpful to plot C versus y_0.] What is the long-term behavior of $P(t)$ in this case?*

Next, we consider the special case when $P(0) = K$, which corresponds to $y_0 = 1$. Notice that the formula for C does not make sense in this case – in fact, the integral (1.4.6) does not make sense when $y = 1$! However, notice that $y(0) = 1$ is an equilibrium solution to (1.4.5), corresponding to the equilibrium solution $P(t) = K$ to (1.4.4).

Activity 1.4.5. *Consider the special when $P(0) = 0$. What are y_0 and C in this case? What is the corresponding solution $y(t)$? Interpret this solution.*

Finally, we consider the case when $P(0) < 0$. While this case is not physically relevant for population modeling, it is interesting from a mathematical point of view. In this situation, we have $-1 < C < 0$. Using (1.4.8) we see that there is some time $t_* > 0$ such that

$$\lim_{t \to t_*^-} y(t) = -\infty,$$

meaning that solutions blow up in finite time.

A mixing problem

We conclude this section by analyzing a mixing problem using separation of variables.

Example 1.4.3. *Suppose we have a 100 gallon tank full of some liquid, in to which a salt water solution is flowing at a rate of 5 gallons per minute. The salt water solution contains 8 ounces of salt per gallon. The salt solution mixes perfectly with the contents of the tank; the resulting mixture is leaving the tank at a rate of 5 gallons per minute.*

Let $S(t)$ be the amount of salt, measured in ounces, in the tank at time t, which we measure in minutes. The differential equation describing this situation is

$$\frac{dS}{dt} = 40 - \frac{1}{20}S.$$

This is a separable equation; the resulting indefinite integral is

$$\int \frac{1}{800 - S}\, dS = \int \frac{1}{20}\, dt.$$

(Note: You don't need to factor out the 1/20, but it is helpful.) Evaluating the integral we find that

$$-\ln(800 - S) = \frac{1}{20}t + C,$$

which we rewrite as

$$S(t) = 800 - Ce^{-t/20}.$$

From this it is easy to see that the solution $S(t)$ exists for all time t and that

$$\lim_{t\to\infty} S(t) = 800.$$

Thus for large times, the amount of salt in the tank approaches 800 ounces.

Notice that we never needed to know what the value of C was! What extra information would we need in order to compute C?

Exercises

Exercise 1.4.1. *Find the general solution of the following equations:*

1. $\dfrac{dy}{dt} = y^3$;
2. $\dfrac{dy}{dt} = (y-1)^2$;
3. $\dfrac{dy}{dt} = e^{-t}(2-y)$;
4. $\dfrac{dy}{dt} = y(3-y)$;
5. $\dfrac{dy}{dt} = y(3-y) - 2$.

Exercise 1.4.2. *A person makes an initial payment of \$3 000 to a retirement fund, and plans to contribute \$6 400 each year continuously for 40 years until the person is 65. The fund will earn 8% a year.*

1. *Find an expression for the value of the fund t years after the initial payment was made;*
2. *What is the value of the fund at age 65;*
3. *Assume that no payments are going to be made after the age 65 and that the same amount of money is going to be taken from the account each month. How big can this sum of money be in order for the retirement fund to last another 30 years?*

Exercise 1.4.3. *Assuming unlimited resources some particular species of fish grows at a steady per capita rate of* 200% *per year. However, it is believed that the lake in which this species lives can only support up to* 10 000 *fish. The lake is private and the owners would like to (continually throughout the year) harvest about* 4 800 *fish. The current fish population in the lake is about* 4 200*. Find an expression for the number of fish as a function of time. Graph your solution. Would you agree that the owner is "overharvesting"?*

Exercise 1.4.4. *A* 100*-gallon mixing vat is initially full of brine in which the concentration of salt is* $0.5\ \frac{lb}{gal}$*. Pure water is pumped into the tank at the rate of* $3\frac{gal}{min}$*. Simultaneously* 3 *gallons of brine per minute are being pumped out. The vat is kept thoroughly mixed at all times.*

1. *Find the expression for the amount (in lb's) of salt in the vat after* t *minutes;*
2. *When will the concentration of salt drop below* $0.1\ \frac{lb}{gal}$*?*

1.5 Propagator functions

In this section we consider initial value problems of the form

$$\frac{dy}{dt} = ry + f \qquad y(t_0) = y_0, \tag{1.5.1}$$

where both r and f are functions of t. We interpret (1.5.1) as a growth model with time-dependent relative growth rate r and forcing term f.

Example 1.5.1. *Suppose an investment is initially valued at* \$5 000 *and is growing with variable annual interest rate* $r(t) = 0.02 + 0.15\cos(t)$*. Suppose furthermore that additional funds are added to the principle investment according to the function* $f(t) = e^{-3t}$*, measured in thousands of dollars per year. Let* $V(t)$ *be the value of the investment fund, measured in thousands of dollars, at time* t*, measured in years. Then growth of* V *is described by*

$$\frac{dV}{dt} = (0.02 + 0.15\cos(t))V + e^{-3t} \qquad V(0) = 4.$$

The differential equation (1.5.1) is not separable. Nevertheless, we are able to obtain reasonable formulas for solutions. To accomplish this, we first consider the initial value problem in the special case that there is no forcing, namely:

$$\frac{dy}{dt} = ry \quad y(t_0) = y_0. \tag{1.5.2}$$

The differential equation in (1.5.2) *is* separable, and thus can be addressed using the methods of the previous section.

Example 1.5.2 (Continuation of Example 1.5.1)**.** *When we ignore forcing, we are left with the IVP*

$$\frac{dV}{dt} = (0.02 + 0.15\cos(t))V \qquad V(0) = 4.$$

This ODE is separable. The solution is given by

$$\begin{aligned} V(t) &= 4\exp\left(\int_0^t (0.02 + 0.15\cos(\tau))d\tau\right) \\ &= 4\exp\left(0.02 + 0.015\sin(t)\right). \end{aligned}$$

We can also arrive at an understanding of the formula for the solution to (1.5.2) by the following line of reasoning. First, in the case that r is constant, the solution to (1.5.2) is

$$y(t) = y_0 \exp\left(r(t - t_0)\right).$$

We rewrite this as

$$y(t) = y_0 \exp\left(\int_{t_0}^{t} r \, d\tau\right). \tag{1.5.3}$$

Suppose now that r is in fact a function. In the case the function given by (1.5.3) is

$$y(t) = y_0 \exp\left(\int_{t_0}^{t} r(\tau) \, d\tau\right). \tag{1.5.4}$$

We can directly compute, using the Fundamental Theorem of Calculus and the chain rule, that

$$\begin{aligned} y'(t) &= \frac{d}{dt}\left[y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right)\right) \\ &= y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right) \frac{d}{dt}\int_{t_0}^{t} r(\tau)\, d\tau \\ &= y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right) r(t) \\ &= r(t)y(t). \end{aligned} \tag{1.5.5}$$

Furthemore

$$y(t_0) = y_0 \exp\left(\int_{t_0}^{t_0} r(\tau)\, d\tau\right) = y_0 \exp(0) = y_0.$$

Thus the function $y(t)$ determined by (1.5.4) is the solution to (1.5.2).

Notice the structure of the formula (1.5.4) – it simply consists of the initial value y_0 multiplied by some function. If we define the function $P(t, s)$ by

$$P(t, s) = \exp\left(\int_{s}^{t} r(\tau)\, d\tau\right), \tag{1.5.6}$$

then the solution to (1.5.2) is given by $y(t) = P(t, t_0)y_0$.

The function $P(t, s)$ given by the formula (1.5.6) is called the ***propagator function*** for the rate function $r(t)$. Multiplication by $P(t, s)$ corresponds to propagation according to the differential equation in (1.5.2), starting at time s and ending at time t.

Propagator functions have the following properties:

1. $P(s, s) = 1$, which we interpret as "propagation from time s to time s does nothing."

2. $\frac{d}{dt}P(t,s) = r(t)P(t,s)$, which we interpret as "the propagator is a solution."

3. $P(t,s) = P(s,t)^{-1}$, which we interpret as "propagation from time t to time s is the inverse operation of propagation from time s to time t.

4. $P(t_2,t_1)P(t_1,t_0) = P(t_2,t_0)$, which we interpret as "propagation from time t_0 to time t_1 and then from time t_1 to time t_2 is the same as propagation from time t_0 to time t_2.

Note that the first two properties imply that $P(t,s)$ is the solution to the initial value problem

$$\frac{d}{dt}P(t,s) = r(t)P(t,s) \qquad P(t_0,s) = 1. \tag{1.5.7}$$

We now show how to use propagator functions in order to find an expression for the solution to the forced equation (1.5.1) with initial condition $y(t_0) = y_0$. First we develop a formula using a intuitive, non-rigorous reasoning; then we verify that the formula solves the desired initial value problem. We begin by supposing that the value of y at time t will be the sum of two terms, one due to the influence of the initial condition and one due to the influence of the forcing. The term due to the initial condition we suppose to be $P(t,t_0)y_0$ (just as in the case when there is no forcing).

The forcing term can be viewed as continuously contributing an "infinitesimal additional amount" to the system. In particular, at a given time s, the quantity $f(s)\,ds$ represents the amount added to the system at that time. (Note that $f(s)\,ds$ has the same units as y.) In order to compute the impact that $f(s)\,ds$ has on the value of $y(t)$ we use the propagator function; the contribution is $P(t,s)f(s)\,ds$. Finally we "add up" all of these infinitesimal contributions, starting with those at time t_0 and ending with those at time t. The result is that we expect the contribution to the value of $y(t)$ due to forcing to be

$$\int_{t_0}^{t} P(t,s)f(s)\,ds$$

and thus we conjecture that the solution to (1.5.1) with initial condition $y(t_0) = y_0$ is

$$y(t) = P(t,t_0)y_0 + \int_{t_0}^{t} P(t,s)f(s)\,ds. \tag{1.5.8}$$

The formula (1.5.8) is called ***Duhamel's formula***.

Let's now verify that the function given by (1.5.8) does indeed solve both (1.5.1) and satisfy the initial condition $y(t_0) = y_0$. The initial condition is straightforward to see, so we focus on the differential equation. It is convenient to use the identity

$$P(t,s) = P(t,t_0)P(t_0,s)$$

in order to rewrite (1.5.8) as

$$y(t) = P(t,t_0)y_0 + P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds.$$

We compute using the properties of propagator functions that

$$\begin{aligned}
y'(t) &= \frac{d}{dt}\left[P(t,t_0)y_0 + P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds\right] \\
&= \frac{d}{dt}\left[P(t,t_0)\right]y_0 \\
&\qquad + \frac{d}{dt}\left[P(t,t_0)\right]\int_{t_0}^{t} P(t_0,s)f(s)\,ds + P(t,t_0)\frac{d}{dt}\int_{t_0}^{t} P(t_0,s)f(s)\,ds \\
&= r(t)P(t,t_0)y_0 + r(t)P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds + P(t,t_0)P(t_0,t)f(t) \\
&= r(t)\left(P(t,t_0)y_0 + P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds\right) + (1)f(t) \\
&= r(t)y(t) + f(t).
\end{aligned}$$

Thus we see that the formula (1.5.8) does indeed satisfy the IVP.

One interesting and important feature of (1.5.8) is that we see explicitly that the solution $y(t)$ is the sum of two parts: the ***homogeneous solution***

$$P(t,t_0)y_0,$$

which due to the initial condition y_0, and the ***particular solution***

$$\int_{t_0}^{t} P(t,s)f(s)\,ds$$

which due to the forcing.

Example 1.5.3. *Let's use propagators to solve the initial value problem*

$$\frac{dy}{dt} = \cos t - 7 \qquad y(0) = \sqrt{17}.$$

In this IVP the rate function is $r(t) = -7$. Thus the propagator is given by

$$P(t,s) = \exp\left(\int_s^t -7d\tau\right) = e^{-7(t-s)}.$$

The initial value $y_0 = \sqrt{17}$ is specified at time $t_0 = 0$. Thus the homogeneous solution is

$$P(t,0)y_0 = \sqrt{17}e^{-7t}.$$

The forcing function is $f(t) = \cos t$; hence the particular solution is

$$\begin{aligned}\int_0^t P(t,s)f(s)ds &= \int_0^t e^{-7(t-s)}\cos(s)\,ds \\ &= \frac{7}{50}\cos(t) - \frac{7}{50}e^{(-7t)} + \frac{1}{50}\sin(t).\end{aligned}$$

Thus the solution to the initial value problem is

$$y(t) = \sqrt{17}e^{-7t} + \frac{7}{50}\cos(t) - \frac{7}{50}e^{(-7t)} + \frac{1}{50}\sin(t).$$

Activity 1.5.1. *Use propagators to solve the initial value problem*

$$\frac{dy}{dt} = 1 - \frac{y}{1-t} \qquad y(0) = 31.$$

Exercises

Exercise 1.5.1. *We arrived at the formula (1.5.4) by "intuiting" what the solution "ought" to be. A more rigorous way to proceed would be to notice that the differential equation in (1.5.2) is separable. Use this fact to directly obtain the formula (1.5.4) by the method of separation of variables.*

Exercise 1.5.2. *Verify (by direct computation) the properties of the propagator function $P(t,s)$:*

1. $P(s,s) = 1$,
2. $\frac{d}{dt}P(t,s) = r(t)P(t,s)$,
3. $P(t,s) = P(s,t)^{-1}$,
4. $P(t_2,t_1)P(t_1,t_0) = P(t_2,t_0)$.

Exercise 1.5.3. *Solve the following IVP using the propagator method:*

$$\frac{dy}{dt} = 2y - t \qquad y(0) = 1.$$

Exercise 1.5.4. *Solve the following IVP using the propagator method:*

$$\frac{dy}{dt} = \frac{y}{1+2t} - 7 \qquad y(0) = 1.$$

Exercise 1.5.5. *Suppose money is invested in a volatile market that has an annual growth rate of* $r(t) = 0.01 + 0.05\cos 10t$*, where* t *is measured in years.*

1. *Make a plot of* $r(t)$ *over a 10 year time period. How should one interpret this growth rate?*
2. *Suppose there is an initial investment of $100. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?*
3. *Suppose instead that no money is initially invested, but that one continuously adds to an investment at a rate of $10 per year. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?*

Exercise 1.5.6. *Solve the initial value problem from Exercise 1.2.4:*

> *We consider a large urn of coffee in cafeteria. A student is draining coffee in to her thermos while at the same time the brewing machine is adding fresh coffee. We want to keep track of the amount of caffeine (in milligrams) in the urn during this process.*
>
> *Suppose the following:*
>
> - *There is initially* 4 *liters of coffee in the urn, with a caffeine concentration of* 100 *mg per liter.*
> - *Starting at time* $t = 0$ *the brewing machine is adding extra strong coffee, which has a concentration of* 250 *mg per liter. This new coffee is being added at a rate of* 25 *mL per minute.*
> - *Starting at time* $t = 0$ *the student begins draining coffee out of the urn at a rate of* 30 *mL per minute.*

Your goal is to find a formula for the amount of caffeine in the urn as a function of time.

1.6 Integrating factors

The method of propagators introduced in the previous section is important in part because it can be generalized to many situations besides those considered in that section, and because it organizes the solution as "response to initial condition" and "response to forcing", an important theoretical perspective. However, for simple initial value problems of the form

$$\frac{dy}{dt} = ry + f \quad y(t_0) = y_0, \tag{1.6.1}$$

where both r and f are functions of t, there is a "trick" – called the ***method of integrating factors*** – that is sometimes useful.

The idea of the trick is reverse engineer the product rule. Recall that for two functions u and y says that

$$\frac{d}{dt}[uy] = u\frac{dy}{dt} + \frac{du}{dt}u.$$

In order to apply this to (1.6.1) we write the differential equation as

$$\frac{dy}{dt} - ry = f.$$

If we multiply this by some function u we obtain

$$u\frac{dy}{dt} + (-ru)y = uf. \tag{1.6.2}$$

The left side of this last equation would look like the derivative of uy if

$$\frac{du}{dt} = -ru. \tag{1.6.3}$$

Let's assume that we have chosen the function u so that it is a solution to (1.6.3). Then (1.6.2) becomes

$$\frac{d}{dt}[uy] = uf.$$

We can integrate this last equation from time t_0 to time t, which leads to

$$u(t)y(t) = u(t_0)y(t_0) + \int_{t_0}^{t} u(\tau)f(\tau)\,d\tau.$$

Dividing by $u(t)$ and using the initial condition $y(t_0) = y_0$ leads to a formula for the solution to (1.6.1), namely

$$y(t) = \frac{u(t_0)}{u(t)} y_0 + \frac{1}{u(t)} \int_{t_0}^{t} u(\tau) f(\tau)\, d\tau. \tag{1.6.4}$$

At this stage, you might think that this looks familiar. . . and it is! Basically, it is the same formula as (1.5.8).

Example 1.6.1. *Consider the differential equation*

$$\frac{dy}{dt} + 2y = 7.$$

Multiplying by u, we can rewrite this equation as

$$u\frac{dy}{dt} + y(-2u) = 7u. \tag{1.6.5}$$

We want u to satisfy

$$\frac{du}{dt} = -2u,$$

which is easy to arrange by choosing $u = e^{-2t}$. Thus (1.6.5) becomes

$$\frac{d}{dt}\left[e^{-2t} y\right] = 7e^{-2t}.$$

We can integrate this in order to obtain

$$e^{-2t} y(t) - y(0) = -\frac{7}{2}\left(e^{-2t} - 1\right).$$

Therefore solutions to the differential equation take the form

$$y(t) = e^{2t} y(0) - \frac{7}{2} + \frac{7}{2} e^{2t}.$$

Activity 1.6.1. *Use the method of integrating factors to find formulas for the following differential equations:*

1. $\dfrac{dy}{dt} + 2ty = t^2$
2. $\dfrac{dy}{dt} + y = \cos t$
3. $\dfrac{dy}{dt} + \cos t\, y = \cos t$

4. $\dfrac{dy}{dt} + y = t^2$

Activity 1.6.2. *Suppose a* $1/2$ *gallon bottle is being filled with water from a rusty pipe. Water coming out of the pipe at one gallon per minute. As the pipe gets flushed out, there is less rust coming out; we assume that the concentration of rust is* $0.01e^{-5t}$ *ounces per gallon. We furthermore assume that the bottle overflows so that it is always full of water and that there is perfect mixing. If there is* 0.05 *ounces of rust in the bottle, find a formula for how much rust is in the bottle as a function of time. At what time is there only* 0.0001 *ounces of rust?*

Exercises

Exercise 1.6.1.

Find the general solution of the following equations:

1. $\dfrac{dy}{dt} = y + t^2$
2. $\dfrac{dy}{dt} = -2ty + e^{-t^2}$
3. $\dfrac{dy}{dt} = -\dfrac{y}{t} + \sin(t)$

Exercise 1.6.2. *Solve the following initial value problems:*

1. $\dfrac{dy}{dt} = y + \sin(t), \quad y(0) = 1$
2. $\dfrac{dy}{dt} = \dfrac{y}{t} + t, \quad y(1) = 0.$

Exercise 1.6.3. *A* 100*-gallon mixing tank is full of pure water at time* $t = 0$. *Salty water of salt concentration* 0.4 lb/gal *is being pumped into the tank at a decreasing rate of* $e^{-0.05t}$ gal/min. *The resulting salty water is also being drained from the tank so that its volume is kept constant at* 100 *gallons. Assuming the tank is always thoroughly mixed, find the amount of salt in the tank (in pounds) at time* t. *What will the concentration of salt roughly become in the long run?*

Exercise 1.6.4. *A* $1\,000$*-gallon tank is full of pure water. Salty water is being pumped into the tank at a decreasing rate of* $\frac{1000}{10+t}$ *gallons per hour;*

here the variable t *denotes the number of hours since the beginning of the mixing process. The concentration of salt in the solution which is pumped into the tank is* 0.01 *pounds per gallon. The tank is constantly being mixed and drained so that the volume of the tank is maintained at* 1 000 *gallons. How much salt will there be in the tank in the long run?*