Chapter 2

Vectors

2.1 Vectors

Key Ideas.

- *Tangent vectors: Given a path $P(t) = (x(t), y(t), \dots)$ the **tangent vector at time t** is*
$$\frac{dP}{dt}(t) = P'(t) = \langle x'(t), y'(t), \dots \rangle.$$
Interpretation:
  - *Geometrically, the tangent vector tell us the instantaneous displacement at time t. Note that the displacement need not be along the path!*
  - *Physically, the tangent vector tells us the velocity at time t.*
- *More generally, a **vector** is a physical displacement.*
If $A = (x_a, y_a, \dots)$ and $B = (x_b, y_b, \dots)$ then $\overrightarrow{AB} = \langle x_b - x_a, y_b - y_a, \dots \rangle$.
Important: a vector by itself does not live at any particular location.
- *Algebraic characterization of vectors:*
  - *adding: $\langle v_1, v_2, \dots \rangle + \langle w_1, w_2, \dots \rangle = \langle v_1 + w_1, v_2 + w_2, \dots \rangle$*
  - *scaling: $\alpha \langle v_1, v_2, \dots \rangle = \langle \alpha v_1, \alpha v_2, \dots \rangle$*

  *Vocabulary: the number α is called a **scalar***
- *Vector notations: brackets, **i**-**j**-**k**, columns.*
$$\langle v_1, v_2 \rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
What do the adding and scaling properties look like in each notation?

- *Lines: a point* $P_0 = (x_0, y_0, \dots)$ *and vector* $\vec{V} = \langle v_1, v_2, \dots \rangle$ *determine a line*

$$P_0 + t\vec{V} = (x_0 + tv_1, y_0 + tv_2, \dots)$$

alternatively

$$\begin{aligned} x(t) &= x_0 + tv_1 \\ y(t) &= y_0 + tv_2 \\ &\vdots \end{aligned}$$

In column notation:

$$\begin{pmatrix} x \\ y \\ \vdots \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ \vdots \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_0 + tv_1 \\ y_0 + tv_2 \\ \vdots \end{pmatrix}$$

- *Application: Suppose we have a path* $P(t)$ *and point* $P_0 = P(t_0)$ *on the path. Let* $\vec{V} = P'(t_0)$ *be the tangent vector at that point. The* ***tangent line*** *at the point* P_0 *is defined by*

$$P_0 + t\vec{V}.$$

Interpretation: the line that best approximates the path at that point.

Exercises

1. For each path below you should

   - draw a picture of the path from $t = 0$ to $t = 1$
   - draw a picture of the tangent vector to the path at time $t = 0$ and at time $t = 1$
   - compute the tangent vector to the path at time $t = 0$ and at time $t = 1$

   (a) $P(t) = (2 + 3t, 5 - t)$

   (b) $P(t) = (\cos(\pi t), 2\sin(\pi t))$

   (c) $P(t) = (e^{2t}, 5e^{2t})$

2. Construct a formula for each path described.

   (a) The path starting at location $(2, 5)$ and having tangent vector $\langle 4, -1 \rangle$.

   (b) The path starting at $(6, 2)$ and having tangent vector $\langle -1, 1 \rangle$.

   (c) The path starting at $(3, 4)$ and having tangent vector $\langle 7, 1 \rangle$.

3. For each of the paths in Problem 1 you should find the tangent line to the path at the location given by $t = 0$. Draw both the path and the tangent line on the same plot.

4. Let $A = (-1, 0, 1)$, $B = (0, 3, 6)$ and $C = (-3, 4, 0)$. Find $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$. Express in all three notations (angle brackets, columns, i-j-k).

5. In this problem let $\vec{a} = \langle 1, -2, 3\rangle$ and $\vec{b} = \langle 2, -3, 4\rangle$. Find the vectors $\vec{a} + \vec{b}$, $-\frac{1}{2}\vec{a}$, $\vec{a} - \vec{b}$ and $-3\vec{a} + 2\vec{b}$.

6. We say that two vectors are ***independent*** if their displacements are not parallel.

   - Example: $\langle 1, 2\rangle$ and $\langle 3, 2\rangle$ are independent.
   - Example: $\langle 1, 2\rangle$ and $\langle 3, 6\rangle$ are *not* independent.

   Let

   $$\vec{u} = \langle 2, -1\rangle \qquad \vec{v} = \langle 1, -2\rangle \qquad \vec{w} = \langle -2, 1\rangle$$

   (a) Are $\vec{u}$ and $\vec{v}$ independent?

   (b) Are $\vec{u}$ and $\vec{w}$ independent?

   (c) Are $\vec{v}$ and $\vec{w}$ independent?

7. Sketch two independent 2D vectors $\vec{a}$ and $\vec{b}$ that share the same basepoint.

   (a) Sketch vectors $2\vec{a}$, $\frac{1}{2}\vec{a} + \frac{3}{2}\vec{b}$, $-\frac{1}{2}\vec{a} + 2\vec{b}$, $\vec{a} - 2\vec{b}$.

   (b) Sketch a different, non-zero 2D vector $\vec{c}$ which also shares a basepoint with $\vec{a}$ and $\vec{b}$. Based on your sketch, estimate the values of α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

   (c) What shape do the vectors $\alpha\vec{a}$ trace out, if the scalar α is

      i. allowed to vary freely throughout the set of all real numbers?

      ii. only allowed to vary between 0 and 1?

      iii. only allowed to vary between -1 and 1?

   (d) What shape do the vectors $\alpha\vec{a} + \beta\vec{b}$ trace out if

      i. α and β are allowed to vary freely throughout the set of all real numbers?

      ii. both α and β are only allowed to vary between 0 and 1?

      iii. only α is allowed to vary freely, but β is limited between -1 and 1?

      iv. only α is allowed to vary freely, but β has to be equal to 1? What is β has to be equal to 2?

2.2 Linear functions

Key Ideas.

- *A point P_0 and a vector $\vec{V}$ determine a line:*
$$P(t) = P_0 + t\vec{V}.$$
- *A point P_0 and two vectors $\vec{U}$, $\vec{V}$ determine a plane:*
$$T(\alpha, \beta) = P_0 + \alpha\vec{U} + \beta\vec{V}$$
- *Matrix notation:*
$$\begin{pmatrix} | & | \\ \vec{U} & \vec{V} \\ | & | \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\vec{U} + \beta\vec{V}$$
In coordinates:
$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} + \beta \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}$$
- *More general matrix multiplication:*
$$\begin{pmatrix} | & | \\ \vec{U} & \vec{V} \\ | & | \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} | & | \\ \alpha\vec{U} + \beta\vec{V} & \gamma\vec{U} + \delta\vec{V} \\ | & | \end{pmatrix}$$
- ***Linear functions*** *take the form*
$$T(\alpha, \beta, \dots) = \begin{pmatrix} | & | & \\ \vec{U} & \vec{V} & \cdots \\ | & | & \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \vdots \end{pmatrix}$$
- ***Affine functions*** *are a linear function plus a shift:*
$$T(\alpha, \beta, \dots) = \begin{pmatrix} | & | & \\ \vec{U} & \vec{V} & \cdots \\ | & | & \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \vdots \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ \vdots \end{pmatrix}$$
- *We draw linear functions with domain/codomain plot. Where do the grid-lines go?*

Exercises.

1. Perform the following matrix multiplications.

$$\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 9 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 9 & 9 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

2. Consider the transformation

$$\begin{aligned} x &= 1 + \alpha + \beta \\ y &= 2 - \alpha + \beta \end{aligned}$$

whose inputs are from the $\alpha\beta$ plane, and whose outputs are in the xy plane.

(a) Write the transformation in matrix notation.

(b) Draw side-by-side the domain ($\alpha\beta$ plane) and the codomain (xy plane).

(c) Indicate where the coordinate grid lines from the domain get mapped by the transformation:

- Start by indicating where the lines $\alpha = 0$ and $\beta = 0$ get mapped.
- Then indicate where $\alpha = 1, 2, -1, -2$ get mapped. Same for $\beta = 1, 2, -1, -2$. Use color coding or font coding to indicate the lines.

(d) The *unit cell* in the domain is the region defined by

$$0 \le \alpha \le 1 \quad \text{and} \quad 0 \le \beta \le 1.$$

Where does the transformation take the unit cell? Indicate by shading in both the domain and the

(e) Give some words that explain what the transformation of this problem is doing to the coordinate grid. Use words like “stretch”, “shift”, “rotate”, etc.

3. Follow the guidelines from Problem 2 to analyze the following. Adjust for the change in dimension when necessary.

(a) $\begin{cases} x = 3\alpha + \beta, \\ y = -1 + \alpha + 3\beta \end{cases}$

(b) $\begin{cases} x = 1 + 2\alpha, \\ y = -3\alpha, \\ z = 4 \end{cases}$

(c) $\begin{cases} x = 2\alpha, \\ y = \frac{1}{3}\beta, \\ z = \alpha + \beta \end{cases}$

(d) $\begin{cases} x = \alpha + \beta + \gamma, \\ y = \beta + \gamma, \\ z = 1 - 2\gamma \end{cases}$

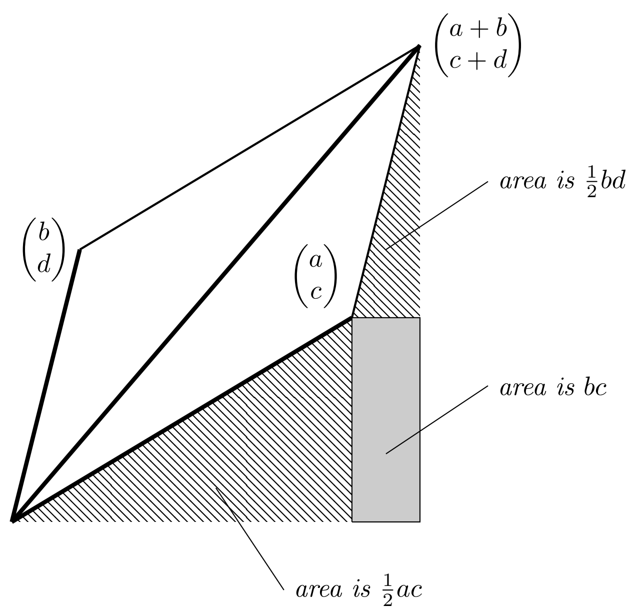
2.3 Determinants

Key Ideas.

- *Question: How does a linear function deform area?*
- *Consider 2D case:*

$$T(\alpha,\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The image of the unit cell is here:



  - *Area of parallelogram is $ad - bc$.*
  - *If we swap the vectors, the sign changes.*

- *The* ***determinant*** *of a 2×2 matrix is defined by*

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

- *The* ***determinant*** *of a 3×3 matrix is defined by*

$$\det \begin{pmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{pmatrix} = a \det \begin{pmatrix} y & z \\ \beta & \gamma \end{pmatrix} - x \det \begin{pmatrix} a & b \\ \beta & \gamma \end{pmatrix} + \alpha \det \begin{pmatrix} a & b \\ y & z \end{pmatrix}$$

- *Geometric interpretation: signed area/volume*

- *Algebraic properties:*
  - *Multiplication property:* $\det(AB) = \det(A)\det(B)$
  - *Transpose property: determinant does not change if we swap rows for columns*
  - *Orientation property: determinant changes sign if we swap adjacent columns (or rows)*
  - *Scaling property: if one column is multiplied by α then the determinant changes by a factor of α*
  - *and many others: take the linear algebra class!*
- *Area Formula 1: If $\vec{V}, \vec{W}$ are 2D vectors then the area of the parallelogram determined by them is*

$$Area = \left| \det \begin{pmatrix} | & | \\ \vec{V} & \vec{W} \\ | & | \end{pmatrix} \right|$$

- *Volume Formula 1: If $\vec{U}, \vec{V}, \vec{W}$ are 3D vectors then the volume of the parallelotope determined by them is*

$$Area = \left| \det \begin{pmatrix} | & | & | \\ \vec{U} & \vec{V} & \vec{W} \\ | & | & | \end{pmatrix} \right|$$

Exercises.

1. Compute the following determinants.

$$\det \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 9 & 9 \end{pmatrix},$$

$$\det \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 4 \\ -1 & 1 & 8 \end{pmatrix}, \qquad \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 2 & 1 \\ 4 & 1 & 4 & 1 \\ -8 & -1 & 8 & 1 \end{pmatrix}.$$

2. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(u, v) = (1 + u + 3v, 1 - u + v).$$

   (a) Express the transformation in matrix notation.

   (b) What is the unit cell $0 \leq u, v \leq 1$ in the uv-plane mapped to under this transformation? (Provide a very rough sketch.)

(c) What is the *stretching factor* of this transformation?

3. Repeat the previour problem for the transformation $T(u, v, w) = (u + v, v + w, w + u)$.

4. Construct transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that stretches the x direction by a factor of 3 and stretches the y direction by a factor of 7. Draw a picture of the transformation. Then determine the area deformation of the transformation.

5. Construct transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that swaps the x and y directions. Draw a picture of the transformation. Then determine the area deformation of the transformation.

6. Construct transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points counterclockwise by an angle of $\pi/3$. Draw a picture of the transformation. Then determine the area deformation of the transformation.

7. Construct transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that takes $(0, 0)$ to $(0, 0)$, takes the x axis to the line through $(1, 2)$, and takes the y axis to the line through $(7, 3)$. Draw a picture of the transformation. Then determine the area deformation of the transformation.

2.4 The dot product

Key Ideas.

- *Definition: For vectors* $\vec{V} = \langle v_1, v_2, \dots \rangle$ *and* $\vec{W} = \langle w_1, w_2, \dots \rangle$ *the dot product is defined by*

$$\vec{V} \cdot \vec{W} = v_1 w_1 + v_2 w_2 + \dots$$

- *Algebraic facts about the dot product:*
  - *Distributive property* $\vec{U} \cdot (\vec{V} + \vec{W}) = \vec{U} \cdot \vec{V} + \vec{U} \cdot \vec{W}$
  - *Commutative property* $\vec{V} \cdot \vec{W} = \vec{W} \cdot \vec{V}$
  - *Associativity of scaling* $\vec{U} \cdot (\alpha \vec{W}) = (\alpha \vec{U}) \cdot \vec{W} = \alpha(\vec{U} \cdot \vec{W})$
- *Geometric features:*
  - *Length/magnitude of a vector* $\|\vec{V}\| = \sqrt{\vec{V} \cdot \vec{V}}$
  - *Unit vector: the unit vector in direction of* $\vec{V}$ *is* $\hat{V} = \frac{1}{\|\vec{V}\|} \vec{V}$.
  - *Angle between vectors* $\vec{V} \cdot \vec{W} = \|\vec{V}\| \|\vec{W}\| \cos\theta$.
  - *Perpendicularity:* $\vec{V} \perp \vec{W}$ *when* $\vec{V} \cdot \vec{W} = 0$.
- *Projections: the projection of* $\vec{W}$ *on to the line determined by* $\vec{V}$ *is given by*

$$\operatorname{proj}_{\vec{V}} \vec{W} = \left(\hat{V} \cdot \vec{W}\right) \hat{V} = \frac{\vec{V} \cdot \vec{W}}{\|\vec{V}\|^2} \vec{V}$$

- *Area Formula 2: the area of the paralellogram determined by* $\vec{V}$ *and* $\vec{W}$ *is*

$$\begin{aligned} Area &= \left| (\vec{V} \cdot \vec{V})(\vec{W} \cdot \vec{W}) - (\vec{V} \cdot \vec{W})^2 \right|^{1/2} \\ &= \left| \det \begin{pmatrix} \vec{V} \cdot \vec{V} & \vec{V} \cdot \vec{W} \\ \vec{V} \cdot \vec{W} & \vec{W} \cdot \vec{W} \end{pmatrix} \right|^{1/2} \end{aligned}$$

  Notes:
  - *This formula works for vectors in any dimension!*
  - *There is an analogous formula for the volume of a parallelotope determined by three vectors.*

Exercises.

1. Consider the points $P = (0, 1, 1)$, $Q = (1, 0, -1)$ and $R = (-1, -1, 1)$.
   (a) Compute the lengths of $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$.
   (b) Compute the angles $\angle PQR$, $\angle QRP$ and $\angle RPQ$. Do they add up to $180°$?
2. Let $\vec{V} = \langle 1, -3 \rangle$ and $\vec{W} = \langle 2, 2 \rangle$.
   (a) Compute $\text{proj}_{\vec{V}} \vec{W}$. Draw a picture illustrating what this computation means geometrically.
   (b) Compute $\text{proj}_{\vec{W}} \vec{V}$. Draw a picture illustrating what this computation means geometrically.
3. Find the following areas and volumes.
   (a) the area of the parallelogram spanned by $\langle 1, 2 \rangle$ and $\langle 2, 1 \rangle$;
   (b) the area of the parallelogram spanned by $\langle -1, 1, 1 \rangle$ and $\langle 2, 1, 0 \rangle$;
   (c) the volume of the parallelotope spanned by $\langle 1, 1, 0 \rangle$, $\langle 1, 0, 1 \rangle$, $\langle 0, 1, 1 \rangle$.
4. Assume that $\hat{a}$, $\hat{b}$ and $\hat{c}$ are three unit vectors forming angles $\frac{\pi}{3}$ with each other.
   (a) What is the area of the parallelogram spanned by $\hat{a}$ and $\hat{b}$?
   (b) What is the area of the parallelogram spanned by $3\hat{a}$ and $8\hat{b}$?
   (c) What is the volume of the parallelotope spanned by $\hat{a}$, $\hat{b}$ and $\hat{c}$?
   (d) What is the volume of the parallelotope spanned by $2\hat{a}$, $3\hat{b}$ and $4\hat{c}$?
   (e) What is the volume of the parallelotope spanned by $\alpha\hat{a}$, $\beta\hat{b}$ and $\gamma\hat{c}$? You may assume that α, β, γ are some positive coefficients.
5. Consider the transformation given by
$$x = 1 + \alpha - 2\beta$$
$$y = 3\alpha + \beta$$
$$z = 2 - \alpha - \beta$$
   What is the area deformation for this transformation?
6. Let $\vec{V} = \langle 1, 2, 3 \rangle$ and $\vec{W} = \langle 1, -1, 1 \rangle$. Can you find a vector $\vec{N}$ such that $\vec{N} \perp \vec{V}$ and $\vec{N} \perp \vec{W}$?

2.5 The cross product

Key Ideas.

- *Motivation: Given vectors* $\vec{V}$ *and* $\vec{W}$ *find a vector perpendicular to both.*
- *Cross product: For vectors* $\vec{V} = \langle v_1, v_2, v_3\rangle$ *and* $\vec{W} = \langle w_1, w_2, w_3\rangle$ *we have*

$$\vec{V} \times \vec{W} = \langle v_2w_3 - v_3w_2, -(v_1w_3 - v_3w_1), v_1w_2 - v_2w_1\rangle$$

- *Compute with determinants:*

$$\begin{aligned}\vec{V} \times \vec{W} &= \det\begin{pmatrix}\mathbf{i} & v_1 & w_1\\ \mathbf{j} & v_2 & w_2\\ \mathbf{k} & v_3 & w_3\end{pmatrix}\\ &= \det\begin{pmatrix}v_2 & w_2\\ v_3 & w_3\end{pmatrix}\mathbf{i} - \det\begin{pmatrix}v_1 & w_1\\ v_3 & w_3\end{pmatrix}\mathbf{j} + \det\begin{pmatrix}v_1 & w_1\\ v_2 & w_2\end{pmatrix}\mathbf{k}\end{aligned}$$

- *Geometric properties:*
  - $\vec{V} \times \vec{W}$ *is perpendicular to* $\vec{V}, \vec{W}$ *according to right-hand rule*
  - *magnitude is area of the parallelogram:* $\|\vec{V} \times \vec{W}\| = \|\vec{V}\|\|\vec{W}\|\sin\theta$.
  - $\vec{V} \times \vec{W} = 0$ *precisely when* $\vec{V}$ *and* $\vec{W}$ *are parallel*
- *Algebraic facts*
  - *Anti-commuting property:* $\vec{V} \times \vec{W} = -\vec{W} \times \vec{V}$
  - *Distributive property:* $\vec{U} \times (\vec{V} + \vec{W}) = \vec{U} \times \vec{V} + \vec{U} \times \vec{W}$
  - *Associativity of scaling:* $(\alpha\vec{V}) \times \vec{W} = \alpha(\vec{V} \times \vec{W}) = \vec{V} \times (\alpha\vec{W})$
- *Planes:*
  - *A plane can be determined by a point* $P_0 = (x_0, y_0, z_0)$ *and a* ***normal vector*** $\vec{N}$. *The equation is*

$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \vec{N} = 0.$$

  - *A plane can be determined by a point* P_0 *and two vectors* $\vec{V}, \vec{W}$ *by setting* $\vec{N} = \vec{V} \times \vec{W}$.

Exercises.

1. Let $\vec{u} = \langle 1, 0, 1\rangle$, $\vec{v} = \langle 1, 1, 0\rangle$ and $\vec{w} = \langle 0, 1, 1\rangle$. Compute and illustrate:
   (a) $\vec{u} \times \vec{v}$;
   (b) $\vec{v} \times \vec{w}$;

(c) $\vec{w} \times \vec{u}$.

2. Show that
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$
Use this to compute the following cross products *without* using determinants.

   (a) $(\mathbf{i} \times \mathbf{j}) \times (\mathbf{k} \times \mathbf{j})$
   (b) $(2\mathbf{k} \times \mathbf{i}) \times (\mathbf{i} \times 3\mathbf{j})$
   (c) $\mathbf{i} \times (\mathbf{j} \times 3\mathbf{i})$
   (d) $(\mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} - \mathbf{j})$
   (e) $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) + \mathbf{j} \times (\mathbf{k} \times \mathbf{i}) + \mathbf{k} \times (\mathbf{i} \times \mathbf{j})$.

3. A point is rotating around the origin within the plane spanned by vectors $\langle 1, 1, 1\rangle$ and $\langle 0, 1, -1\rangle$. Find the axis of rotation, that is, find the line around which the point is rotating.

4. Consider the plane through the point $(1, 0, 2)$ spanned by the vectors $\langle 0, 1, 1\rangle$ and $\langle -1, 0, 1\rangle$. Write the equation for this plane in the form of $\_\_\_x + \_\_\_y + \_\_\_z = \_\_\_$.

5. Describe the plane $2x + 5y + 7z = 10$ in the following ways

   (a) by specifying the x, y, z intercepts
   (b) by specifying a point on the plane and the normal vector to the plane
   (c) by specifying a point on the plane and two vectors tangent to the plane.

6. Consider the line in the xy-plane passing though the point $(2, 3)$ spanned by the vector $\langle 5, 8\rangle$.

   (a) Find a normal vector to this line.
   (b) Use the normal vector you found to obtain the equation of the line in the form of $\_\_\_x + \_\_\_y = \_\_\_$.

7. Describe the line $2x + 3y = 10$ in the following ways:

   (a) by giving the x and y intercepts
   (b) by specifying a point on the line and a normal vector to the line
   (c) by specifying a point on the line and a tangent vector to the line

2.6 Vector fields

Key Ideas.

- *A* ***vector field*** *gives us a vector at each location:*

$$\vec{V}(x, y, \dots) = \langle v_1(x, y, \dots), v_2(x, y, \dots), \dots \rangle$$

  Physical examples include wind/fluid flow, electric/magnetic fields, gravitational fields, etc.

- *Coordinate vector fields*
  - *Geometric idea: tangent vectors to motion along "curvilinear coordinate grids"*
  - *Computation requires* ***partial derivative:*** $\frac{\partial}{\partial u}$ *means take derivative where u is the variable and all other variables are considered constants.*
  - *If we have the transformation*

$$T(u, v) = (x(u, v), y(u, v), \dots)$$

    then the coordinate vector fields are

$$\partial_u T = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \dots \right\rangle$$

  - *Notation: write* ∂_u *instead of* $\partial_u T$ *when the transformation is clear*
  - *Careful when plotting: the transformation tells us the location!*

- *Example: polar coordinates. The vectors*

$$\partial_r = \langle \cos\theta, \sin\theta \rangle \qquad \partial_\theta = \langle -r\sin\theta, r\cos\theta \rangle$$

  are located at $(x, y) = (r\cos\theta, r\sin\theta)$ *Draw a picture!*

Exercises.

1. Draw by hand a plot the following vector fields. Try first without using technology; then use technology to check your picture.

   (a) $\vec{V} = \langle 1, 4 \rangle$

   (b) $\vec{W} = \langle x, y \rangle$

   (c) $\vec{U} = \langle y, x \rangle$

   (d) $\vec{P} = \langle 1, x^2 \rangle$

   (e) $\vec{B} = \langle x, -y \rangle$

   (f) $\vec{A} = \langle -y, x \rangle$

2. Use "technology" to plot the vector fields $\vec{F} = \langle -y, x, 0 \rangle$ and $\vec{G} = \langle x, y, z \rangle$. Please include the print-out of your work. Then describe in words what each vector field looks like.

3. Let (r, θ, z) be the standard cylindrical coordinates in space.
   (a) Find formulas for the vector fields ∂_r, ∂_θ and ∂_z.
   (b) Specifically, evaluate and sketch the vector fields ∂_r, ∂_θ and ∂_z at the points with Cartesian coordinates $(1, 0, 0)$ and $(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$.
   (c) Complete a rough sketch of the vector fields ∂_r, ∂_θ and ∂_z.
4. Let (r, θ, ϕ) be the standard spherical coordinates in space.
   (a) Find formulas for vector fields ∂_r, ∂_θ and ∂_ϕ.
   (b) Specifically, evaluate and sketch the vector fields ∂_r, ∂_θ and ∂_ϕ at the points with Cartesian coordinates $(1, 0, 0)$ and $(0, \frac{\sqrt{3}}{2}, \frac{1}{2})$.
   (c) Complete a rough sketch of the vector fields ∂_r, ∂_θ and ∂_ϕ.
5. Consider the transformation given by

$$\begin{aligned} x &= u + v \\ y &= u - v \end{aligned}$$

   Compute the coordinate vector fields ∂_u and ∂_v. Plot these vector fields.
6. Consider the transformation given by

$$\begin{aligned} x &= u^2 - v^2 \\ y &= 2uv, \end{aligned}$$

   where we restrict attention to $u \geq 0$. Compute the coordinate vector fields ∂_u and ∂_v. Plot these vector fields.
7. Construct a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T(0, 0) = (0, 0)$ and is such that the coordinate vector fields are

$$\partial_u = \langle 2, 2 \rangle \qquad \partial_v = \langle 1, -3 \rangle.$$

   Make a domain/codomain drawing of the transformation.