Project: Waves with quadratic potential

In this project you explore a variation of the wave equation. The spatial domain is $\mathbb{R}$, and we impose the "infinite string" boundary conditions at $x = \pm\infty$. However, we don't use the usual expression for the potential energy. Instead, we use

$$V(u) = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + x^2u^2\right\}dx.$$

This expression for the potential energy means that it is "more costly" for the wave to be located far away from $x = 0$. The total energy is

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + x^2u^2\right\}dx.$$

1. Show that energy is conserved if u satisfies the modified wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - x^2u. \qquad \text{(SHO-W)}$$ SHO-wave

2. Now go looking for standing wave solutions of the form $u(t,x) = A(t)\psi(x)$. You should get a differential equation for A and a differential equation for ψ, both of which involve an eigenvalue λ. Use integration by parts (♡!) to show that $\lambda \leq 0$ and thus that we can write $\lambda = -\omega^2$.

3. It turns out that the differential equation for ψ is famous. To see this we make a change of variables, writing $\psi = e^{-x^2/2}H$ for some function H. Show that H must satisfy the equation

$$\frac{d^2H}{dx^2} - 2x\frac{dH}{dx} + 2\kappa H = 0 \qquad \text{(H)}$$ Hermite

   where $\kappa = -\frac{\lambda+1}{2}$.

   Then show that your friends the Hermite polynomials H_0, H_1, H_2, H_3 each satisfy (H) for some value of κ. What is the value of κ corresponding to each solution?

4. Using the previous problem, you can now construct four standing wave solutions to (SHO-W). Make a plot of each of these solutions... and use Desmos to animate them. What is the energy of each solution?

5. We cannot be satisfied with having only four standing wave solutions – we expect an infinite list! To get this list, we return to (H). Our plan is to look for more polynomial solutions.

   A generic polynomial takes the form

   $$H(x) = \sum_{k=0}^{??} a_k x^k.$$

   Plug this in to (H) and show that the coefficients must satisfy the recurrence relation

   $$a_{k+2} = \frac{2(k-\kappa)}{(k+2)(k+1)} a_k.$$

   From this we deduce two things:

   - All of the even coefficients are determined by a_0 and all of the odd coefficients are determined by a_1.
   - If κ is an even integer, then the list of non-zero even coefficient is finite; if κ is an odd integer, then the list of nonzero odd coefficients is zero.

   To display these properties, do the following:

   - Show that if $a_0 \neq 0$ and $a_1 = 0$ then we recover the solution is $a_0 H_2(x)$, provided we choose $\kappa = 2$.
   - Show that if $a_0 = 0$ and $a_1 \neq 0$ then we recover the solution is $a_1 H_3(x)$, provided we choose $\kappa = 3$.

   How should you construct more Hermite polynomials? What are the corresponding standing wave solutions to (SHO-W)?

6. From the previous problem we have a procedure to construct an infinite list of polynomial solutions to (H). These give us an infinite list of eigenfunctions ψ_k. What are the corresponding eigenvalues? How do we know that these eigenfunctions will be orthogonal to one another?