Project: Fourier series and the periodic heat equation

In this project you explore the heat equation on the domain $[-L, L]$ with periodic boundary conditions.

1. We take the perspective that a function $u(t, x)$ is a path of vectors in $\mathcal{L}^2([-L, L])$, meaning that for each time t we have a vector. With this perspective, the little Fourier transform takes $u(t, x)$ to a path in $l^2(\mathbb{Z})$, meaning that for each t we have a sequence. Show that

$$f\left(\frac{\partial u}{\partial t}\right)_k = \frac{d}{dt} f(u)_k.$$

   and that

$$f\left(\frac{\partial u}{\partial x}\right)_k = i\frac{k\pi}{L} f(u)_k.$$

   We interpret this to mean that the little Fourier transform does not touch t differentiation, and that the little transform takes spatial differentiation to multiplication by $i\frac{k\pi}{L}$.

   Test out the second formula above with the function $u(x) = x$. What is $f(u)$? What is $f(u')$? Does the formula work?

2. The formulas above allow us to transform the heat equation in to an infinite list of ODEs. Show that the heat equation becomes

$$\frac{d}{dt} f(u)_k = -\left(\frac{k\pi}{L}\right)^2 f(u)_k.$$

   (It might be helpful to use the notation $\alpha = f(u)$.)

   We assume that the solution u to the heat equation has some initial heat profile $u(0, x) = u_0$. How does this initial condition transform?

3. The infinite list of ODE should be easily solvable. Show that the solution is
$$f(u)_k = f(u_0)_k G_k.$$
where $G_k = e^{-\left(\frac{k\pi}{L}\right)^2 t}$.

   Notice that this is a product! Thus we expect this to come from a convolution in physical space…but a convolution of what? We need a function $g(t, x)$ such that $f(g)_k = G_k$.

4. Use the Fourier inversion (aka Fourier series) to construct the desired function g. Use Desmos to plot $g(t, x)$ for various values of t. What does this function look like? I claim that this is the periodic version of the function K that appeared in the previous project. Do you agree?

5. We now can construct the periodic solution to the heat equation in two ways.
   - Since we know $f(u)_k$ we can directly compute the Fourier series.
   - Or we can view the solution as $u = u_0 * g$, where u_0 is the initial shape and g is the thing we constructed above.

   Explain how to interpret each of these perspectives from a physical point of view.

6. Choose some "interesting" initial shape u_0 and construct the Fourier series solution to the heat equation. Make some plots. Make some intelligent remarks. What do you think?