Hermite stuff

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1 Hermite stuff

We define the Hermite inner product by

$$\langle u, v\rangle_H = \int_{-\infty}^{\infty} u(x)v(x)e^{-x^2}\, dx.$$

The first four Hermite polynomials are

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x. \end{aligned}$$

We can show that they are orthogonal to one another by computing the Hermite inner products.

First, note that by symmetry we have

$$\int_{-\infty}^{\infty} (\text{any odd function}) e^{-x^2} dx = 0.$$

Thus automaticaly we find

$$\begin{aligned} \langle H_0, H_1\rangle_H &= 0 \qquad \langle H_0, H_3\rangle_H = 0 \\ \langle H_2, H_1\rangle_H &= 0 \qquad \langle H_2, H_3\rangle_H = 0 \end{aligned}$$

For the rest of the integrals, we first note that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

```
In [28]: integrate(e^(-x^2),x,-infinity, infinity)
```

```
Out[28]: sqrt(pi)
```

Next we note that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \int_{-\infty}^{\infty} \underbrace{x}_{f} \underbrace{x e^{-x^2}\, dx}_{dg} = \underbrace{x}_{f} \underbrace{-\frac{1}{2} e^{-x^2}\, dx}_{g} \Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$$

In [29]:
```
integrate((x^2)*e^(-x^2),x,-infinity, infinity)
```

Out[29]:
```
1/2*sqrt(pi)
```

Similarly

$$\int_{-\infty}^{\infty} x^4 e^{-x^2}\,dx = \frac{3}{4}\sqrt{\pi}$$

In [30]:
```
integrate((x^4)*e^(-x^2),x,-infinity, infinity)
```

Out[30]:
```
3/4*sqrt(pi)
```

Thus we compute

$$\langle H_0, H_2\rangle = \int_{-\infty}^{\infty} (1)(4x^2-2)dx = 4\left(\frac{1}{2}\sqrt{\pi}\right) - 2\left(\sqrt{\pi}\right) = 0$$
$$\langle H_1, H_3\rangle = \int_{-\infty}^{\infty} (2x)(8x^3-12x)dx = 16\left(\frac{3}{4}\sqrt{\pi}\right) - 24\left(\frac{1}{2}\sqrt{\pi}\right) = 0$$

Of course we could have had the computer do this for us:

In [31]:
```
# define Hermite polynomials and the Hermite weight function
var('x')
H0(x)=1
H1(x) = 2*x
H2(x) = 4*x^2 -2
H3(x) = 8*x^3 - 12*x
w(x) = e^(-x^2)
```

In [32]:
```
# compute inner product
integrate( H1(x)*H3(x)*w(x) ,x,-infinity, infinity)
```

Out[32]:
```
0
```

We can also compute the norms of the Hermite polynomials. We do this with the identity $\|u\|_H^2 = \langle u, u\rangle_H$.

$$\|H_0\|_H^2 = \sqrt{\pi}$$
$$\|H_1\|_H^2 = 2\sqrt{\pi}$$
$$\|H_2\|_H^2 = 8\sqrt{\pi}$$
$$\|H_3\|_H^2 = 48\sqrt{\pi}$$

In [33]:
```
latex(integrate( H3(x)*H3(x)*w(x) ,x,-infinity, infinity))
```

Out[33]:
```
48 \, \sqrt{\pi}
```

And yes, I did indeed have Sage give me the LaTeX code directly. Cause that's a thing.

2 Approximating $f(x) = e^x$

Let's consider the function $f(x) = e^x$. First we show that $\|f\|_H$ is finite. To do this by hand involves completing the square:

$$\langle f, f\rangle_H = \int_{-\infty}^{\infty} e^{2x-x^2}\, dx = \int_{-\infty}^{\infty} e^{-(x-1)^2+1}\, dx = e\int_{-\infty}^{\infty} e^{-(x-1)^2}\, dx = e\sqrt{\pi}$$

Let's check this with the machine:

```
In [34]: f(x) = e^x
         latex(integrate( f(x)*f(x)*w(x) ,x,-infinity, infinity))
```

```
Out[34]: \sqrt{\pi} e
```

The machine agrees, although I take exception to the decision to put $\sqrt{\pi}$ before e.
Now we compute the coefficients

$$\alpha_k = \frac{\langle H_k, f\rangle_H}{\|H_k\|_H^2}$$

Using the machine, I find:

$$\alpha_0 = e^{\frac{1}{4}}$$
$$\alpha_1 = \frac{1}{2}\, e^{\frac{1}{4}}$$
$$\alpha_2 = \frac{1}{8}\, e^{\frac{1}{4}}$$
$$\alpha_3 = \frac{1}{48}\, e^{\frac{1}{4}}$$

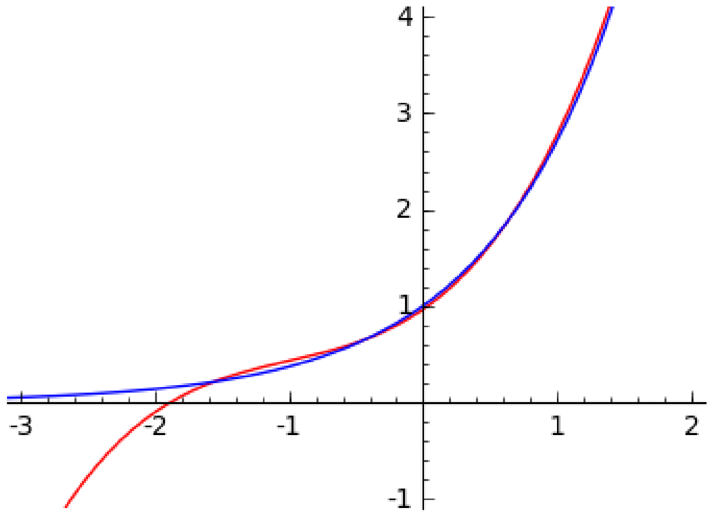
```
In [38]: H(x) = H3(x)
         latex(integrate( H(x)*f(x)*w(x) ,x,-infinity, infinity)/integrate( H(x)*H(x)*w(x) ,x,-i
```

```
Out[38]: \frac{1}{48} \, e^{\frac{1}{4}}
```

Here is a plot of the function f in blue and the approximation in red. Not bad!

```
In [53]: fH(x) = (e^(1/4))*H0(x) + (1/2*e^(1/4))*H1(x) + (1/8*e^(1/4))*H2(x)+ (1/48*e^(1/4))*H3(

         fHplot = plot(fH(x),(x,-10,10), color = 'red')
         fplot = plot(f(x), (x,-10, 10))
         mainplot = fHplot + fplot
         mainplot.show(xmin=-3, xmax = 2, ymin = -1,ymax = 4,figsize = [4,3])
```



In [ ]: