Chapter 6

Wave equation in 2D and 3D

6.1 Derivation of the wave equation in two dimensions

We now turn our attention to the wave equation on domains with more than one dimension. Roughly speaking, a region Ω is ***n*-dimensional** if it can be parametrized by n coordinate functions. Thus the interior of the square and the disk are both two-dimensional regions. The solid ball is three-dimensional, while the surface of the sphere is two-dimensional.

In this part of the course we discuss the wave equation in two-dimensional domains, focusing on the square and disk. The methods used can be extended to other domains in other dimensions. The reason that we focus on these two examples is that the details become more complicated for more complicated domains. But the main principles remain the same.

Suppose Ω is a bounded two-dimensional domain with boundary $\partial\Omega$. Our immediate goal is to define the wave equation for the domain Ω. We consider functions $u(t, \mathbf{x})$, where $\mathbf{x}$ represents a point in Ω. As in Chapter **??**, we define the kinetic energy of u to be

$$K[u] = \frac{1}{2} \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dA.$$

Recall that in one dimension the potential energy of a function described the extent to which the function was non-constant. Thus in higher dimensions it makes sense to define the potential energy of the function u to be

$$V[u] = \frac{1}{2} \int_{\Omega} \| \operatorname{grad} u \|^2 \, dA.$$

We thus define the total energy of function u, at time t, to be

$$E[u] = K[u] + V[u] = \frac{1}{2}\int_{\Omega}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \|\operatorname{grad} u\|^2\right) dA.$$

We can compute the time-derivative of the energy $E[u]$ by

$$\frac{d}{dt}E[u] = \int_{\Omega}\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \operatorname{grad} u \cdot \operatorname{grad}\left(\frac{\partial u}{\partial t}\right)\right) dA$$

We now make use of the identity (see Exercise 6.1.1)

$$\operatorname{grad} u \cdot \operatorname{grad}\left(\frac{\partial u}{\partial t}\right) = \operatorname{div}\left(\frac{\partial u}{\partial t}\operatorname{grad} u\right) - \frac{\partial u}{\partial t}\operatorname{div}(\operatorname{grad} u)$$

to express the time derivative of energy as

$$\frac{d}{dt}E[u] = \int_{\Omega}\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}\operatorname{div}(\operatorname{grad} u) + \operatorname{div}\left(\frac{\partial u}{\partial t}\operatorname{grad} u\right)\right) dA.$$

Applying the Divergence Theorem to the final term yields

$$\frac{d}{dt}E[u] = \int_{\Omega}\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\operatorname{grad} u)\right) dA + \int_{\partial\Omega}\frac{\partial u}{\partial t}\operatorname{grad} u \cdot \widehat{\mathbf{n}}\, ds, \qquad (6.1)$$ general-energy-der

where $\widehat{\mathbf{n}}$ be the outward-pointing unit normal to $\partial\Omega$.

We now define two different possible boundary conditions.

- A function $u(t, \mathbf{x})$ with domain Ω satisfies the ***Dirichlet boundary condition on $\boldsymbol{\Omega}$*** if $u(\mathbf{x}) = 0$ for all points $\mathbf{x}$ on the boundary $\partial\Omega$ and at all times t.
- A function $u(t, \mathbf{x})$ with domain Ω satisfies the ***Neumann boundary condition on $\boldsymbol{\Omega}$*** if $\operatorname{grad} u(\mathbf{x}) \cdot \widehat{\mathbf{n}} = 0$ for all points $\mathbf{x}$ on the boundary $\partial\Omega$ and at all times t.

Under either the Dirichlet or the Neumann boundary condition, the last integral in (6.1) vanishes. Thus we see that both the Dirchlet and Neumann settings, the energy $E[u]$ is conserved provided u satisfies

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\operatorname{grad} u) = 0. \qquad (6.2)$$ 2D:first-wave

The equation (6.2) is called the ***wave equation***, and is a natural generalization of the equation (2.7) derived in Chapter **??**. The operator $\operatorname{div}(\operatorname{grad} \cdot)$

is called the ***Laplacian*** (or “Laplace operator”) and is given the symbol Δ. Thus the wave equation is also written

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0. \tag{6.3}$$

2D:wave

It is convenient to express the wave equation (6.3) in various coordinate systems.

Example 6.1.1

In Cartesian coordinates (x, y) is straightforward to compute

$$\Delta u = \operatorname{div}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Thus the wave equation (6.3) is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \tag{6.4}$$

2D:cartesian-wave

Example 6.1.2

In order to express the wave equation (6.3) in polar coordinates we make use of the chain rule, together with the relations

$$x = r\cos\theta \qquad y = r\sin\theta$$

to obtain the transformations

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$
$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Inserting these in to the Cartesian expression yields

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

Thus the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \qquad (6.5)$$ 2D:polar-wave

HW:divergence-identity **Exercise 6.1.1.** *Suppose f is a function and $\mathbf{v}$ is a vector field.*

1. *Show that* $\operatorname{div}(f\mathbf{v}) = (\operatorname{grad} f)\cdot\mathbf{v} + f\operatorname{div}(\mathbf{v})$. *(Hint: Work in Cartesian coordinates.) Explain why it makes sense to think of this as a "vector calculus product rule."*

2. *Use the vector calculus product rule and the Divergence Theorem to show that*

$$\int_\Omega (\operatorname{grad} f)\cdot\mathbf{v}\,dA = \int_{\partial\Omega} f(\mathbf{v}\cdot\widehat{\mathbf{n}})\,dA - \int_\Omega f\operatorname{div}(\mathbf{v})\,dA. \qquad (6.6)$$ vector-IBP

   Explain why we can think of this identity as "vector calculus integration-by-parts."

3. *Use the integration by parts formula above to show that*

$$\int_\Omega (\operatorname{grad} f)\cdot(\operatorname{grad} g)\,dA = \int_{\partial\Omega} f(\operatorname{grad} g\cdot\widehat{\mathbf{n}})\,ds - \int_\Omega f(\Delta g)\,dA. \qquad (6.7)$$

Exercise 6.1.2.

1. *Suppose Ω is the region in the plane determined by*

$$0 \le x \le L \quad \text{and} \quad 0 \le y \le L,$$

   where L is some positive number. We refer to this region as the ***square of size L****. Make a sketch of the region and label the places where $x = 0$, $x = L$, $y = 0$, and $y = L$.*

   Consider a function $u(t, x, y)$ defined for times $t \ge 0$ and for (x, y) in the square of size L. Express Dirichlet boundary conditions for u in terms of the Cartesian coordinates (x, y).

2. *Suppose Ω is the region in the plane determined, in polar coordinates, by*

$$0 \le r \le 1 \quad \text{and} \quad -\pi \le \theta \le \pi.$$

We refer to this region as the ***unit disk****. Make a sketch of the region and label the places where $r = 0$, $r = 1$, $\theta = -\pi$, and $\theta = \pi$.*

Consider a function $u(t, r, \theta)$ defined for times $t \geq 0$ and for points (r, θ) in the unit disk. Express Dirichlet boundary conditions on unit disk in terms of polar coordinates (r, θ).

What condition must hold at $r = 0$? What condition must hold at $\theta = \pm\pi$?

Exercise 6.1.3.

1. *Express the energy $E[u]$ in terms of Cartesian coordinates. The take the time derivative of $E[u]$ in order to derive the wave equation while working always in Cartesian coordinates.*

2. *Express the energy $E[u]$ in terms of polar coordinates. The take the time derivative of $E[u]$ in order to derive the wave equation while working always in polar coordinates.*

6.2 Standing waves in two dimensions

::standing-wave-2D

Just as in the case of the one-dimensional wave equation, our approach to studying the wave equation in two dimensions is centered on the idea of standing wave solutions. For simplicity, we focus on the case of Dirichlet boundary conditions.

Standing wave solutions take the form

$$u(t, \mathbf{x}) = A(t)\psi(\mathbf{x}). \tag{6.8}$$ 2D:standing-wave-ansatz

Inserting this expression in to the wave equation (6.3) we see that we must have

$$\frac{1}{A}\frac{d^2A}{dt^2} = \frac{1}{\psi}\Delta\psi.$$

Since the left side depends only on time t and the right side depends only on the spatial variables $\mathbf{x}$, we deduce that standing wave solutions arise from solutions A, ψ to

$$\frac{d^2A}{dt^2} = \lambda A \qquad \Delta\psi = \lambda\psi,$$

where λ is some constant.

In order for a function u of the form (6.8) to satisfy the Dirichlet boundary condition for all times t, it must be the case that ψ satisfies the Dirichlet boundary condition. Therefore, can find standing wave solutions by first finding a function ψ and number λ satisfying

$$\Delta\psi = \lambda\psi \text{ on } \Omega \qquad \psi = 0 \text{ on } \partial\Omega, \tag{6.9}$$ 2D:Dirichlet-eigen

and then finding a function A satisfying

$$\frac{d^2A}{dt^2} = \lambda A. \tag{6.10}$$ 2D:amplitude-equat

Once we know the value of λ it is easy to find solutions to (6.10). Thus the real difficulty is finding ψ and λ that satisfy (6.9). The problem of finding ψ and λ that satisfy (6.9) is called the ***Dirichlet eigenvalue problem for Ω***. The functions ψ that solve (6.9) are called the ***Dirichlet eigenfunctions*** for the region Ω and the corresponding values of λ are called the ***Dirichlet eigenvalues (or "Dirichlet spectrum")*** of the region Ω.

In general, finding the eigenfunctions and eigenvalues of a region Ω is a difficult (and interesting) task. In most cases there are not explicit formulas for the eigenfunctions. However, if the region has special symmetries, then it is possible to use some algebraic tricks to obtain formulas for the eigenfunctions and associated eigenvalues. The next two chapters of the course work through the details in the case of where Ω is a square and the case where Ω is a disk.

• **needed**

- obtain eigenfunctions that are orthogonal (see HW for special case)
- thus we can do series solutions just as before

Exercise 6.2.1. *Just as in the case of the one-dimensional wave equation, we can easily show that the eigenvalue λ present in (6.9) must be negative. Do this by first showing that any solution ψ, λ to (6.9) must satisfy*

$$\int_\Omega \psi\Delta\psi\, dA = \lambda\psi^2\, dA.$$

Then use the last identity from Exercise 6.1.1, together with the Dirichlet boundary condition, to show that

$$-\int_\Omega \|\operatorname{grad}\psi\|^2\, dA = \lambda\psi^2\, dA.$$

Conclude that $\lambda = -\omega^2$ for some $\omega \geq 0$.

(As a bonus, show that if $\lambda = 0$ implies $\psi = 0$, and thus that $\omega > 0$.)

• **needed**

Exercise 6.2.2. *Show that if ψ_1, ψ_2 are eigenfunctions with eigenvalues $\mu_1 \neq \mu_2$, then $\langle\psi_1, \psi_2\rangle = 0$.*

6.3 Three dimensions

Thus far, this chapter has discussed the wave equation in two dimensions. However, the story is essentially unchanged if we want to consider the wave equation in three (or more) spatial dimensions.

Notationally, when working in three spatial dimensions the various integrals over the domain Ω must involve with the appropriate volume form dV rather than the area form dA. Likewise, integrals over the boundary $\partial\Omega$ are now area integrals rather than line integrals. Otherwise, the discussion in the previous sections all hold unchanged.

Exercise 6.3.1. *Write down the wave equation in cylindrical coordinates.*

Exercise 6.3.2. *Write down the wave equation in the spherical coordinates*

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi.$$

Exercise 6.3.3. *Suppose we want to write down a wave equation where the domain is the two-dimensional surface of the sphere. We can proceed as follows.*

- *First, write down the expression for* $\|\operatorname{grad} u\|^2$ *in spherical coordinates.*
- *Second, we assume that* u *does not depend on the radial function* r *(because* r *is constant on the sphere), thus we remove any terms involving* $\frac{\partial}{\partial r}$.
- *Third, we assume that the sphere is of radius one, and thus we set* $r = 1$ *everywhere.*

When combining all this with the area element for the sphere, you should find that the potential energy is

$$V[u] = \frac{1}{2}\int_0^\pi \int_{-\pi}^\pi \left(\left(\frac{\partial u}{\partial \phi}\right)^2 + \frac{1}{\sin^2\phi}\left(\frac{\partial u}{\partial \theta}\right)^2 \right) \sin\phi \, d\theta \, d\phi.$$

Now use this to obtain an expression for the wave equation.

6.4 Homework for 2019

Exercise 6.4.1. *Suppose that ψ_1 and ψ_2 are Dirichlet eigenfunctions on domain Ω with eigenvalues λ_1 and λ_2. This means that*

$$\Delta\psi_1 = \lambda_1\psi_1 \text{ on } \Omega, \qquad \psi_1 = 0 \text{ on } \partial\Omega$$

and

$$\Delta\psi_2 = \lambda_2\psi_2 \text{ on } \Omega, \qquad \psi_2 = 0 \text{ on } \partial\Omega.$$

Show that if $\lambda_1 \neq \lambda_2$ then the inner product

$$\langle\psi_1, \psi_2\rangle = \int_\Omega \psi_1\psi_2 dA$$

is zero. [Hint: we did the one-dimensional computation in class. Also: ♡.]

Exercise 6.4.2. *In class we found that there is an infinite list of Dirichlet eigenfunctions on the rectangle $\Omega = [0, L] \times [0, M]$. They are*

$$\psi_{kl}(x, y) = \sin\left(\frac{k\pi}{L}\right)\sin\left(\frac{l\pi}{M}\right).$$

The corresponding eigenvalues are

$$\lambda_{kl} = -\left(\left(\frac{k\pi}{L}\right)^2 + \left(\frac{l\pi}{M}\right)^2\right).$$

1. *Compute the "potential energy" $V(\psi_{kl})$ of each eigenfunction. What is the relationship between the potential energy and the eigenvalue λ_{lk}?*

2. *Consider now the special case of the unit square, where $L = 1$ and $M = 1$. Find a way to list the eigenfunctions in increasing order of potential energy. What can you say about solutions with the same energy?*

Exercise 6.4.3. *I'd like you to make some plots of the Dirichlet eigenfunctions ψ_{kl}. Since Desmos doesn't do 3D plots very well, let's use Sage.*

Here is the code I'm using

```
var("x","y")
k=2
l=5

psi(x,y) = sin(k*pi*x)*sin(l*pi*y)
```

```
graph = plot3d(psi(x,y), (x,0,1), (y,0,1),adaptive=True,
    color=['red','yellow'])

contour = contour_plot(psi(x,y), (x,0,1),(y,0,1), fill=false
    , contours=20)

graph.show()
#contour.show()
```

A link is here. You can comment out the graph and display the contour plot. Play around with these plots until you get a sense of these eigenfunctions. See also the reference page for contour plots and the reference page for 3d plots.

Exercise 6.4.4. *Suppose we have a function u that is defined on the rectangle $\Omega = [0, L] \times [0, M]$. Can we approximate this function by a combination of the Dirichlet eigenfunctions? In this problem we address this question with the following:*

1. *We use the inner product*

$$\langle u, v \rangle = \iint_\Omega uv \, dA.$$

Show that the eigenfunctions are orthogonal to one another, meaning that

$$\langle \psi_{kl}, \psi_{ij} \rangle = 0$$

unless $k = i$ and $l = j$.

2. *Compute $\|\psi_{kl}\|^2$.*

3. *How should one choose the constants α_{ij} so that the sum*

$$\sum_{i,j} \alpha_{ij} \psi_{ij} \tag{6.11}$$

best approximates the function u?

4. *Consider the function $u(x, y) = 1 + 2x + 3y$. Compute the coefficients α_{ij}. Make a plot of the corresponding approximation. How good is the approximation?*

5. *Explain how to use all this to construct solutions to the wave equation.*