Chapter 5

Fourier transform

The little Fourier transform is used to study functions with domain $[-L, L]$. We now develop tools for studying functions with domain all of $\mathbb{R}$. Our approach is to take the theory for $\mathcal{L}^2([-L, L])$ and take the limit as $L \to \infty$. Before we get in to technical details, let's do a rough (or "formal") calculation.

Suppose we have a function u that is in $\mathcal{L}^2([-L, L])$ for any $L > 0$. The formula (4.4) means that for most values of x we have

$$u(x) = \sum_{k=-\infty}^{\infty} f(u)_k e^{i\frac{k\pi}{L}x} = \sum_{k=-\infty}^{\infty} \left(\frac{f(u)_k}{\pi/L}\right) e^{i\frac{k\pi}{L}x} \frac{\pi}{L}. \tag{5.1}$$ motivate-FT

We interpret the term in round brackets as follows. The numerator $f(u)_k$ represents how much of frequency $k\frac{\pi}{L}$ is present in the function u, while the denominator π/L represents a single "unit" of frequency. Thus the term in round brackets can be viewed as the "amount of frequency per unit frequency".

As L gets very large, the gap π/L between the possible frequencies shrinks towards zero. So it makes sense to think of frequency as a continuous variable rather than a discrete variable. Let ξ be the continuous frequency variable. In the limit $L \to \infty$ we have $\pi/L \to d\xi$ and thus $k\pi/L \to \xi$. Using this, together with the formula for the little Fourier transform, we see that in the limit $L \to \infty$ we have

$$\frac{f(u)_k}{\pi/L} = \frac{1}{2\pi}\int_{-L}^{L} u(y) e^{-i\frac{k\pi}{L}y}\, dy \to \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y) e^{-i\xi y}\, dy,$$

This last expression is a function of ξ, which we again interpret as the "normalized" amount of frequency ξ present in the function u.

The discussion in the previous paragraph motivates us to define the "big" ***Fourier transform*** of a function u to be the function $\widehat{u}(\xi)$ defined by

$$\widehat{u}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y)e^{-i\xi y}\,dy. \tag{5.2}$$ FT-formula

The Fourier transform $\widehat{u}(\xi)$ tells us how much of frequency ξ is present in a u having domain $\mathbb{R}$. You should think of this as analogous to the Fourier coefficient c_k telling us how much of frequency $k\pi/L$ is in a function u on the domain $[-L, L]$. The key idea is that since u is defined on all of $\mathbb{R}$ and not on an interval of finite width, all frequencies are possible!

We continue our formal calculation as follows. If we discretize the ξ axis by tiny intervals of width $\Delta\xi = \frac{\pi}{L}$, then from (5.1) we have

$$u(x) = \sum_{k=-\infty}^{\infty}\left(\frac{f(u)_k}{\pi/L}\right)e^{i\frac{k\pi}{L}x}\frac{\pi}{L} = \sum_{k=-\infty}^{\infty}\widehat{u}(k\Delta\xi)e^{i(k\Delta\xi)x}\Delta\xi.$$

In the limit as $L \to \infty$ (and thus $\Delta\xi \to 0$), the right hand side is precisely the Riemann sum for the integral

$$\int_{-\infty}^{\infty}\widehat{u}(\xi)e^{i\xi x}\,d\xi.$$

Thus we expect that, for a reasonably nice functions u, to have the formula

$$u(x) = \int_{-\infty}^{\infty}\widehat{u}(\xi)e^{i\xi x}\,d\xi. \tag{5.3}$$ FT-recovery

We should interpret this last equation to mean that we can construct the function u by summing over all frequencies ξ the amount $\widehat{u}(\xi)$ of that frequency present times the periodic function $e^{i\xi x}$ having that frequency. This is analogous to the formula coming from the Fourier series.

We conclude this section with the remark that all of the formal calculations done above can be massaged into proper mathematical statements. However, doing so requires some technicalities that are beyond the scope of this course. You are encouraged to take the complex variables course, and also the real analysis course, before returning to these issues.

HW:FT-unit-pulse **Exercise 5.0.1.**

1. *Find a constant C so that the function*

$$u(x) = \begin{cases} C & \text{if } |x| \le a \\ 0 & \text{otherwise} \end{cases}$$

has norm $\|u\| = 1$.

2. *Compute the Fourier transform of* u.

3. *What happens to* $\widehat{u}$ *as* $a \to \infty$*? What happens as* $a \to 0$*?*

Exercise 5.0.2. *Compute the Fourier transform of the following functions.*

1. $u(x) = e^{-|x|}$

2. $u(x) = e^{-ax^2}$*, where* a *is some positive constant. [Hint: Complete the square.]*

5.1 Properties of the Fourier transform

In this section we study the Fourier transform from the perspective of a transformation from one inner product space to another. The main point is that the Fourier transform has properties analogous to the properties of the little Fourier transform.

First we define $\mathcal{L}^2(\mathbb{R})$ to be the vector space of all piecewise-smooth functions u with domain $\mathbb{R}$ such that

$$\|u\|^2 = \int_{-\infty}^{\infty} |u(x)|^2 \, dx < \infty.$$

The function u may be real-valued or complex valued. Notice that such functions must decay to zero as $x \to \pm\infty$. Thus whenever we compute using integration by parts we may disregard the boundary term[1].

For such functions we define an inner product by

$$\langle u, v \rangle = \int_{-\infty}^{\infty} \overline{u(x)} \, v(x) \, dx.$$

It is easy to see that this makes $\mathcal{L}^2(\mathbb{R})$ in to a complex inner product space.

The Fourier transform can be viewed as a linear transformation from $\mathcal{L}^2(\mathbb{R})$ to $\mathcal{L}^2(\mathbb{R})$, where the input of the transformation is a function u and the output is $\widehat{u}$. We use the symbol $\mathcal{F}$ for this transformation; thus $\widehat{u} = \mathcal{F}[u]$. Thus (5.2) can also be written

$$\mathcal{F}[u](\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) e^{-i\xi y} \, dy. \qquad (5.4)$$ FT-define

Remark 5.1.1. *There are several conventions about where to put the factor of* 2π *in the definition of the Fourier transform. When reading a paper or book it is a good idea to check the convention being used.*

[1] Technically, we are using the fact that the Schwartz class is dense in $L^2(\mathbb{R})$.

We now take a look at several properties of the Fourier transform. The first two properties are linearity and invertibility. It is easy to see from the formula (5.4) that the Fourier transform is ***linear***, by which we mean that

$$\mathcal{F}[\alpha u + \beta v] = \alpha \mathcal{F}[u] + \beta \mathcal{F}[v]$$

for any functions u, v in $\mathcal{L}^2(\mathbb{R})$ and scalars α, β.

The formula (5.3) says that if we define

$$\mathcal{F}^{-1}[\widehat{u}](x) = \int_{-\infty}^{\infty} \widehat{u}(\xi) e^{i\xi x} d\xi \qquad (5.5)$$

FT-inverse

then

$$\mathcal{F}^{-1}[\mathcal{F}[u]] = u.$$

Thus we see that the Fourier transform is ***invertible***.

The next property is that the Fourier transform is an ***isomorphism***, meaning that the norm of $\mathcal{F}[u]$ is a multiple of the norm of u. To see this we compute, using (5.4) and (5.5), that

$$\begin{aligned}
\|\widehat{u}\|^2 &= \langle \widehat{u}, \widehat{u} \rangle \\
&= \int_{-\infty}^{\infty} \overline{\widehat{u}(\xi)} \widehat{u}(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} \overline{\widehat{u}(\xi)} \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) e^{-i\xi y}\, dy\, d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) \int_{-\infty}^{\infty} \overline{\widehat{u}(\xi) e^{i\xi y}} d\xi\, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) \overline{\mathcal{F}^{-1}[\widehat{u}](y)}\, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) \overline{u(y)}\, dy \\
&= \frac{1}{2\pi} \|u\|^2.
\end{aligned}$$

Example 5.1.2

ex:FT-triangle

Consider the triangle function

$$u(x) = \begin{cases} 1 + x & \text{if } -1 \le x \le 0 \\ 1 - x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

We compute the Fourier transform of u to be

$$\begin{aligned}\widehat{u}(\xi) &= \frac{1}{2\pi}\int_{-1}^{0}(1+x)e^{-ix\xi}\,dx + \frac{1}{2\pi}\int_{0}^{1}(1-x)e^{-ix\xi}\,dx \\ &= \frac{2(1-\cos(\xi))}{\xi^2}.\end{aligned}$$

The Sage code used to compute this is:

```
var('x,z')
show(integral( (1+x)*exp(-I*x*z), (x,-1,0))
     + integral( (1-x)*exp(-I*x*z), (x,0,1)))
```

(Notice that we had to do some simplification by hand after using Sage!) Even though it appears that $\widehat{u}(\xi)$ is not defined at zero, we can see (using Taylor series) that the function $\widehat{u}$ is actually defined there, with $\widehat{u}(0) = 1$.

It is easy to compute $\|u\|^2 = 2/3$. We can have Sage compute $\|\widehat{u}\|^2$, verifying the isomorphism property that $\|\widehat{u}\|^2 = 2\pi\|u\|^2 = 4\pi/3$. The Sage code is

```
var('x,z')
show(integral((2*(1-cos(z))/z^2)^2,(z,-infinity,
    infinity)))
```

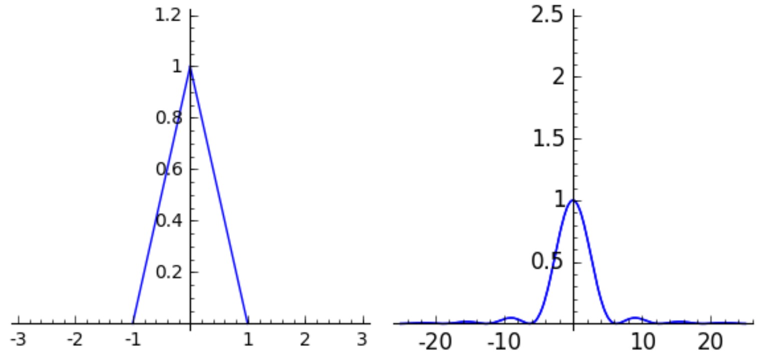
We now explore how the Fourier transform $\widehat{u}$ changes when we change the function u. We consider two types of changes: scaling and translation. First we consider scaling. Fix a function u and define a function v by $v(x) = u(ax)$, where $a > 0$ is some constant. Graphically, the function v takes u and stretches it horizontally by a factor of $1/a$. (You can see this by observing that $v(1) = u(a)$. Computing using change of variables we see that

$$\widehat{v}(\xi) = \frac{1}{a}\widehat{u}(\xi/a).$$

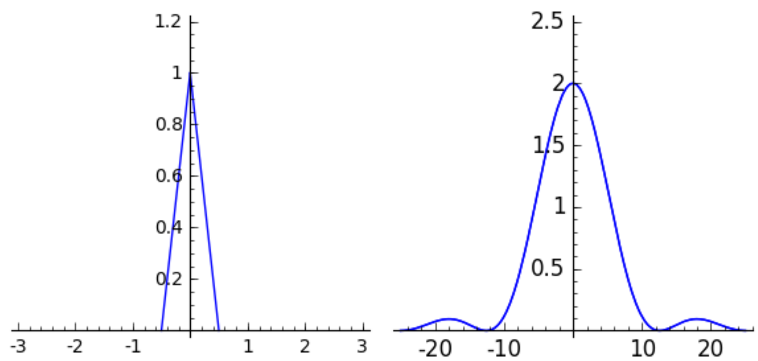
We interpret this as follows: If a function u is stretched horizontally by a factor of $1/a$, then the Fourier transform is stretched horizontally by a factor of a and vertically by a factor of $1/a$.

Example 5.1.3

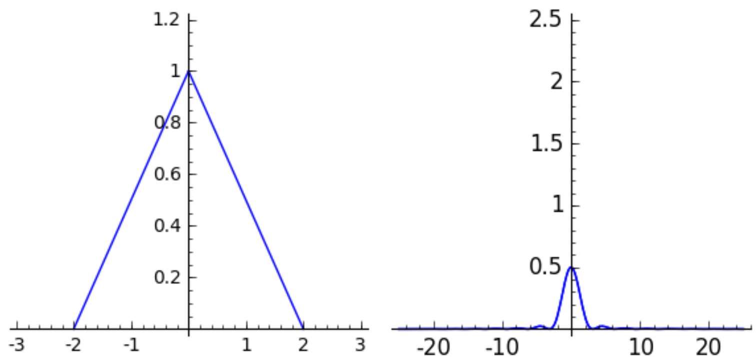
Consider the triangle function u defined in Example 5.1.2. The following image shows the plot of u on the right and the plot of $\widehat{u}$ on the left.



We now stretch u by a factor of $1/2$. The following image shows a plot of $u(2x)$ on the left and the plot of $\frac{1}{2}\widehat{u}(\xi/2)$ on the right. Notice that the height of the Fourier transform has increased, and the spread has also increased.



Finally, we stretch u by a factor of 2. The following image shows a plot of $u(x/2)$ on the left and the plot of $2\widehat{u}(2\xi)$ on the right. Notice that the height of the Fourier transform has decreased, and the spread has also decreased.



From this example we see how concentrating a function leads to its Fourier transform becoming more spread out, and vice versa.

The Sage code used to generate the above images is the following.

```
var('x,z')
a = 1/2

uplot = plot( (1+a*x), (x,-1/a,0)) + plot( 1-a*x, (x
    ,0,1/a)) + plot(1.2, (x,-3,3),thickness=0)

hatu(z) = 2*(1-cos(z))/z^2
hatuplot = plot( a*hatu(z/a) , (z,-25,25))+ plot(2.5, (x
    ,-3,3),thickness=0)

mainplot = graphics_array((uplot,hatuplot))
show(mainplot, figsize = [6,3])
```

We now consider the effects of translating a function. Fix a function u and define a function v by $v(x) = u(x-b)$ where b is some real number. Graphically, the function v takes u and shifts it to the right by amount b. (If b is negative then the shift is to the left.) We can compute (see Exercise 5.1.1)

$$\widehat{v}(\xi) = e^{-ib\xi}\widehat{u}(\xi). \tag{5.6}$$

FT-phase-shift

The last property of the Fourier transform we discuss is the multiplication property. Recall that in the case of the little Fourier transform, pointwise multiplication of the transformed functions corresponded to the convolution product of the original functions. The same is true for the big Fourier transform. For two functions u and v in $\mathcal{L}^2(\mathbb{R})$ we define the ***con-***

folution product to be the function $u * v$ defined by

$$(u * v)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y)v(x-y)\,dy.$$

We claim that

$$\mathcal{F}[u * v] = \mathcal{F}[u] \cdot \mathcal{F}[v]. \tag{5.7}$$ FT-convolution

The verification of this fact is the task of Exercise 5.1.2.

HW:verify-phase-shift **Exercise 5.1.1.** *Verify the formula (5.6). Hint: compute directly using the definition (5.4).*

HW:FT-convolution **Exercise 5.1.2.** *Verify the formula (5.7). Hint: compute directly using the definition (5.4), changing the order of integration, and changing variables.*

HW:FT-derivative-property **Exercise 5.1.3.** *Verify the* ***derivative property*** *for Fourier transform:*

$$\mathcal{F}[u'](\xi) = -i\xi\mathcal{F}[u](\xi). \tag{5.8}$$

5.2 Fourier transform and the wave equation

We now use the Fourier transform to address the initial value problem for the wave equation on the real line. That is, we seek a function $u(t,x)$ that is defined for $t \geq 0$ and all real numbers x satisfying the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

satisfying the initial conditions

$$u(0,x) = u_0(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(0,x) = v_0(x)$$

for some given functions u_0 and v_0, and satisfying the infinite string boundary condition from (2.1):

$$u(t,x) \to 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,x) \to 0 \qquad \text{as} \quad x \to \pm\infty.$$

Our approach is the apply the Fourier transform to the entire problem. Using Exercise 5.1.3 we see that the transform of wave equation is

$$\frac{d^2\widehat{u}}{dt^2}(t,\xi) = -\xi^2\widehat{u}(t,\xi). \tag{5.9}$$ FT-wave

This is paired with the initial conditions

$$\widehat{u}(0,\xi) = \widehat{u}_0(\xi) \qquad \text{and} \qquad \frac{d\widehat{u}}{dt}(0,\xi) = \widehat{v}_0(\xi). \tag{5.10}$$ FT-wave-ic

This is fantastic – the Fourier transform has converted the wave equation into an ordinary differential equation (for each frequency ξ).

Using the methods from the differential equations course we see that the general solution to (5.9) is

$$\widehat{u}(t,\xi) = \alpha\cos(\xi t) + \beta\sin(\xi t).$$

Applying the initial conditions (5.10), we see that the solution to the transformed initial value problem is

$$\widehat{u}(t,\xi) = \widehat{u}_0(\xi)\cos(\xi t) + \widehat{v}_0(\xi)\frac{\sin(\xi t)}{\xi}.$$

It now remains to transform this back to the physical space variables.

We first focus on the term $\widehat{u}_0(\xi)\cos(\xi t)$, which we write as

$$\widehat{u}_0(\xi)\cos(\xi t) = \frac{1}{2}\left(e^{it\xi}\widehat{u}_0(\xi) + e^{-it\xi}\widehat{u}_0(\xi)\right).$$

Notice that each of the terms in the parentheses take the form (5.6). Thus we see that

$$\widehat{u}_0(\xi)\cos(\xi t) = \frac{1}{2}\left(\mathcal{F}[\widehat{u}_0(x+t)] + \mathcal{F}[\widehat{u}_0(x-t)]\right).$$

In other words

$$\mathcal{F}^{-1}[\widehat{u}_0(\xi)\cos(\xi t)] = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right). \tag{5.11}$$ FT-wave-shape-term

We interpret this last expression as follows. Suppose we have initial conditions where $v_0 = 0$. Then the solution to the wave equation on the line consists of two traveling waves, each having the shape as the half-sized initial condition, with one wave moving to the left and one moving to the right. At time $t = 0$ the two waves line up perfectly to form the initial shape of u_0.

Example 5.2.1

Suppose our initial conditions are given by

$$u_0(x) = e^{-x^2} \qquad \text{and} \qquad v_0(x) = 0.$$

Then the corresponding solution to the initial value problem is

$$u(t,x) = \frac{1}{2}\left(e^{-(x+t)^2} + e^{-(x-t)^2}\right),$$

which we can view as the left-moving traveling wave

$$\frac{1}{2}e^{-(x+t)^2}$$

combined with the right-moving wave

$$\frac{1}{2}e^{-(x-t)^2}.$$

We now address the term

$$\widehat{v}_0(\xi)\frac{\sin(\xi t)}{\xi}.$$

Notice that

$$\widehat{v}_0(\xi)\frac{\sin(\xi t)}{\xi} = \frac{1}{2}\int_{-t}^{t} \widehat{v}_0(\xi)\cos(\xi\tau)\,d\tau.$$

Thus we can use the same technique as above in order to express this as the time-integral of a left-moving translation and a right-moving translation. In Exercise 5.2.1 you show that the result is that

$$\mathcal{F}^{-1}\left[\widehat{v}_0(\xi)\frac{\sin(\xi t)}{\xi}\right](t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_0(z)\,dz. \tag{5.12}$$

FT-wave-velocity-t

Combining (5.12) with (5.11) we see that the solution to the entire initial value problem is

$$u(t,x) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} v_0(z)\,dz.$$

Notice that at location x and time t the value of the solution depends on the initial velocity over the interval $[x-t, x+t]$ and the value of the initial shape at the edge of that interval. This fact, known as *Huygen's principle*, captures the finite speed of propagation for the wave equation.

HW:FT-wave-velocity-term

Exercise 5.2.1. *Verify the formula* (5.12).

Exercise 5.2.2. *Find the solution to the wave equation with initial conditions*

$$u_0(x) = 0 \quad \text{and} \quad v_0(x) = \begin{cases} 1 & \text{if } |x| \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

For which values of t, x *is the solution* $u(t, x) = 0$*?*