Project: The one-dimensional heat equation

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Physically we can think of the unknown $u(t, x)$ as being the temperature (along a thin wire) at location x and time t. We assume that one of the four boundary conditions holds.

1. Explain how we can also interpret the heat equation as the gradient flow for the potential energy

$$V(u) = \frac{1}{2} \int_{\text{start}}^{\text{end}} \left(\frac{\partial u}{\partial x} \right)^2 \, dx.$$

2. Show that if u is a solution to the heat equation then we have

$$\frac{d}{dt} V(u) = - \int_{\text{start}}^{\text{end}} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \, dx.$$

   From this, conclude that the heat equation does reduce the size of the potential energy along the flow.

3. Show that the heat equation is linear, meaning that the superposition principle holds.

4. Suppose that u is a solution to the heat equation. The "total amount of heat" at time t is represented by the integral

$$H(u) = \int_{\text{start}}^{\text{end}} u(t, x) \, dx.$$

For which boundary conditions is the total amount of heat constant in time? If you can, use this to give a physical interpretation (in terms of heat/temperature) of the four different boundary conditions.

5. Consider the function
$$K(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}},$$
which is defined for $-\infty < x < \infty$.
    - Show that K is a solution to the heat equation.
    - Draw a picture of the graph of K at various times $t > 0$. What is the behavior of this solution?
    - What is the total amount of heat of the solution K?
    - Can you give a physical interpretation of this solution?
6. Now assume that $0 \le x \le L$. At $x = 0, L$ we enforce the Neumann boundary condition; physically we interpret this boundary condition to mean that no heat flows in/out the ends of the wire. Find all of the (Neumann) eigensolutions to the heat equation. How do these solutions behave as time evolves. Give a physical interpretation of the behavior.
7. Compute the potential energy of each of the Neumann eigensolutions. Does the potential energy tend to zero?