Chapter 4

A motivating example

-series-hypothesis

At the end of the previous chapter we saw that addressing the initial value problem using eigensolutions depending on answering the following question:

> Given a function f defined on $[0, L]$, can we choose constants a_k so that the functions f_n, defined by
>
> $$f_n(x) = \sum_{k=1}^{n} \alpha_k \psi_k(x) \tag{4.1}$$
>
> Fourier-series-hypothesis
>
> converge to f as $n \to \infty$?

The mathematics needed to address these questions is based on the work pioneered by Jean-Baptiste Joseph Fourier. In 1822 Fourier published his book *Théorie analytique de la chaleur* (Analytic theory of heat), in which he claimed that any "reasonable" function can be constructed as the infinite sum of sine functions.

There is a fair amount of mathematical theory that we need to develop before we are able to understand Fourier's claim in detail. Before we do that, however, it might be helpful to look at an example from Fourier himself. In §228 of his book[1], Fourier presents the example of a function $\varphi(x)$ defined for $0 \leq x \leq \pi$ by

$$\varphi(x) = \begin{cases} x & \text{for } 0 \leq x \leq \alpha, \\ \alpha & \text{for } \alpha \leq x \leq \pi - \alpha, \\ \pi - x & \text{for } \pi - \alpha \leq x \leq \pi. \end{cases} \tag{4.2}$$

Fourier-phi

[1] A digital copy of Fourier's book is freely available at Google books. I encourage you to take a look!

Fourier then states that

$$\varphi(x) = \frac{4}{\pi}\left(\sin(\alpha)\sin(x) + \frac{\sin(3\alpha)}{3^2}\sin(3x) + \frac{\sin(5\alpha)}{5^2}\sin(5x) + \frac{\sin(7\alpha)}{7^2}\sin(7x) + \dots\right) \quad (4.3)$$

Fourier-phi-series

The following Sage code produces a plot of the function above with $\alpha = \pi/3$ and the sum truncated at the $\sin(7x)$ term.

```
var('x,k,n')
n=3
a = pi/3

f(x)=(4/pi)*sum(sin((2*k+1)*a)*sin((2*k+1)*x)/((2*k+1)^2),k
    ,0,n)

plot(f,(x,0,pi),figsize=[4,2])
```

If we take n to be a larger and larger, then we see that the function

$$\varphi_n(x) = \sum_{k=0}^{n} \frac{\sin((2k+1)\alpha)}{(2k+1)^2}\sin((2k+1)x)$$

does seem to approach the function $\varphi(x)$.

• revise?

Exercise 4.0.1. *Take α very close to zero. You should see some weird wiggles appearing in the function φ_n. This is called "Gibbs phenomenon." Look it up on Wikipedia.*

• needed

Exercise 4.0.2. $\alpha = \pi/2$ *This is the initial shape of a plucked string. What is the corresponding solution to the wave equation?*