Chapter 4

Fourier series

4.1 The complex inner product space $l^2(\mathbb{Z})$

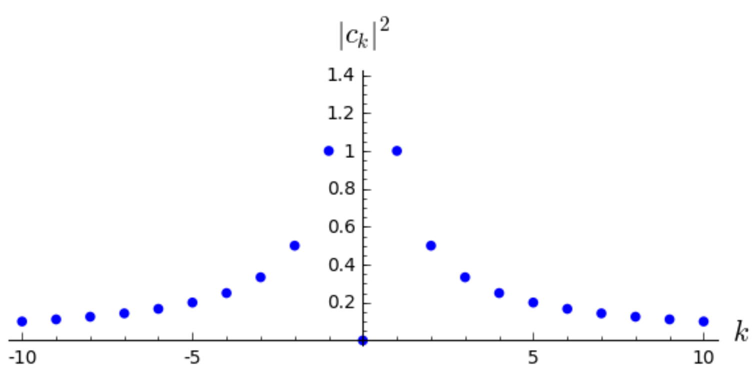
The (periodic) Fourier transform takes a function u in $\mathcal{L}^2([-L, L])$ and gives out an infinite list c_k of complex numbers. In order to study this process, we view a list of numbers c_k as a function where the inputs are integers and the outputs are complex numbers. We use the phrase "sequence" to describe these lists, but we think of them as functions with domain $\mathbb{Z}$.

For instance, in Example 3.5.1 we computed the Fourier coefficients for the function $u(x) = x$ to be the sequence

$$c_k = \begin{cases} (-1)^k \frac{L}{k\pi} i & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases} \tag{4.1}$$

LFT-line

We visualize this sequence by plotting $|c_k|$ versus k as follows. (For the purposes of plotting, we set $L = \pi$.)



The Sage code used to generate this plot is the following.

```
# counter k ranges from -n to n
n=10

# create a collection of points called "data"
data = [(k, k) for k in [-n..n]]

# fill in the values for negative k
for k in [-n..-1]:
    data[k+n] = (k,-1/k)

# fill in the values for positive k
for k in [1..n]:
    data[k+n] = (k,1/k)

# fill in value for k=0
data[n] = (0,0)

# display plot
list_plot(data,figsize=[6,3], size=30,axes_labels=("$k$","$|
    c_k|^2$"),ymax=1.4)
```

We define a complex inner product of two sequences $a = \{a_k\}$ and $b = \{b_k\}$ by

$$\langle a, b\rangle = \sum_{k=-\infty}^{\infty} \overline{a_k} b_k.$$

Using this inner product we define the norm of a sequence by

$$\|a\|^2 = \langle a, a\rangle = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

We define the vector space $l^2(\mathbb{Z})$ to be the collection of all sequences with finite norm. This vector space is called ***little el two***, though it is typically written symbolically and not with the name spelled out like this.

The sequence $c = \{c_k\}$ given by is in $l^2(\mathbb{Z})$ with norm given by

$$\|c\|^2 = \sum_{k\neq 0} \frac{L^2}{\pi^2 k^2} = \frac{2L^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{L^2}{3}.$$

This was computed using the Sage code

```
var('k')
show(sum(1/k^2, k,1,infinity))
```

Exercise 4.1.1. *Let c be the sequence of Fourier coefficients arising from the function $u(x) = x^2$, as computed in Example 3.5.2.*

1. *Make a plot to visualize the function c.*
2. *Compute $\|c\|^2$. How does this norm compare to the norm of u in $\mathcal{L}^2([-L, L])$? (They should differ by a factor of $2L$. Do they?)*

4.2 Fourier series as a linear transformation

The process of taking a function u in $\mathcal{L}^2([-L, L])$ and producing a sequence in $l^2(\mathbb{Z})$ can be viewed as a function, which we call f (for "Fourier"). The domain of f is $\mathcal{L}^2([-L, L])$, meaning that the inputs to the function f are themselves functions. The codomain is $l^2(\mathbb{Z})$, meaning that the outputs are also functions; in fact, the outputs are sequences.

Notice that both the domain and codomain of f are vector spaces. A function where both the domain and codomain are vector spaces is called a ***transformation***. Our particular transformation f we call the ***little Fourier transform***. We call it "little" because the outputs are in "little l two."

Let's introduce some notation. Given a function u in $\mathcal{L}^2([-L, L])$, we denote by $f(u)$ the sequence of Fourier coefficients. The k^{th} number in the sequence $f(u)$ we denote by $f(u)_k$. For example, if the function u is given by the formula $u(x) = x$, then $f(u)$ is the sequence given by the formula

$$f(u)_k = \begin{cases} (-1)^k \frac{L}{k\pi} i & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

More generally, we have a formula for $f(u)$ that comes from (3.8):

$$f(u)_k = \frac{1}{2L} \int_{-L}^{L} e^{-i\frac{k\pi}{L}y} u(y)\, dy. \tag{4.2}$$ LFT-forward

Example 4.2.1

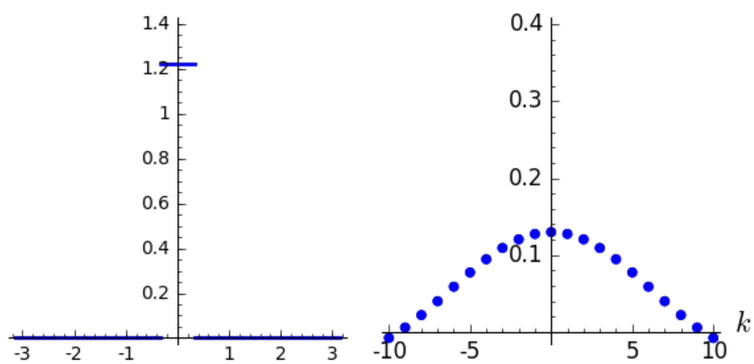
ple:normalized-pulse For any positive number $a < L$ let u be the function in $\mathcal{L}^2([-L, L])$ given by

$$u(x) = \begin{cases} 0 & \text{if } |x| > a \\ \frac{1}{\sqrt{2a}} & \text{if } |x| \leq a. \end{cases}$$

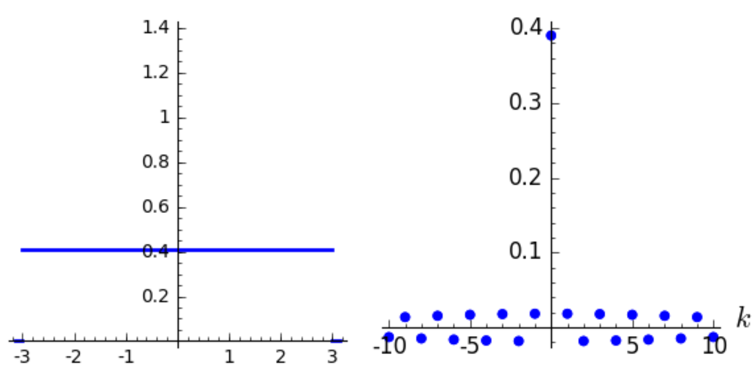
We call the function u the ***normalized pulse*** because $\|u\| = 1$. We compute the little Fourier transform of u to be given by

$$f(u)_k = \begin{cases} \dfrac{1}{\sqrt{2a}\,k\pi} \sin\left(\dfrac{k\pi a}{L}\right) & \text{if } k \neq 0 \\ \\ \dfrac{\sqrt{2a}}{2L} & \text{if } k = 0. \end{cases}$$

Since $f(u)$ is real-valued, we can plot $f(u)_k$ versus k. For the purposes of plotting, set $L = \pi$. When $a = 1/3$, the plots of u and $f(u)$ are the following.



When $a = 3$ the plots of u and $f(u)$ are the following.



The Sage code used to generate this second plot is the following.

```
# counter k ranges from -n to n
n=10
```

```
a=1/2

# hack define the normalized function
u1(x) = 1/sqrt(2*a)
u0(x) = 0

# create a collection of points called "data"
data = [(k, k) for k in [-n..n]]

# fill in the values for negative k
for k in [-n..-1]:
    data[k+n] = (k,sin(k*a)/(sqrt(2*a)*k*pi) )

# fill in the values for positive k
for k in [1..n]:
    data[k+n] = (k,sin(k*a)/(sqrt(2*a)*k*pi))

# fill in value for k=0
data[n] = (0,sqrt(2*a)/(2*pi))

# construct plots
xplot = plot(u1, (x,-a, a), thickness=2) + plot(u0, (x,-
    pi, -a), thickness=2)+ plot(u0, (x,a, pi), thickness
    =2) + plot(1.4,(x,0,1), thickness=0)
kplot = list_plot(data, size=30,axes_labels=("$k$",""),
    ymax = .4)

# display both plots side-by-side
mainplot = graphics_array((xplot,kplot))
show(mainplot, figsize = [6,3])
```

Notice that when a is small, then the pulse $u(x)$ is concentrated near $x = 0$, while values of k for which the transform $f(u)_k$ is (relatively) larger is spread out. Conversely, when a is large, then the pulse $u(x)$ is spread out, while the values of k for which $f(u)_k$ is larger is more concentrated. Note also the different vertical scales on the two plots.

Remark 4.2.2. *The previous example illustrates a more general feature of Fourier transforms.*

- *When k is small, the coefficients $f(u)_k$ tell us about the large scale structure of the function u. Thus if $f(u)_k$ is large when k is small, then there are large-scale structures present in the function u.*

  Recall that the frequency of the standing waves is dependent on the size of k, and that small values of k correspond to slower frequency oscillations. Thus that part of u determined by $f(u)_k$ with k small is often called the "low frequency part" of the function u.

- *Conversely, the coefficients $f(u)_k$ with k large correspond to fine-scale structure in the function u. If $f(u)_k$ is large when k is relatively large, then u has fine-scale structure present.*

  Since larger values of k correspond to standing waves with faster oscillations, the part of u determined by $f(u)_k$ with k large is called the "high frequency part" of the function u.

Of course, there isn't really a sharp cutoff between "small" and "large" values of k, so the distinctions between high and low frequency are a bit arbitrary. But they are still a useful way to talk about a function.

The interpretation of $f(u)_k$ as the large and small frequency parts has an important application in physics and engineering. If you only care about some length scales (or frequency scales) below some threshold, then you can simply set $f(u)_k$ to zero for k larger than the corresponding threshold. This gives an approximation of u that is "good enough" for the situation at hand.

Finally, note that in this interpretation of the little Fourier transform the trigonometric functions $\cos\left(\frac{k\pi}{L}x\right)$ and $\sin\left(\frac{k\pi}{L}x\right)$ are considered to be "purely" at a single length scale because they are themselves the shapes of a single standing wave with periodic boundary conditions.

We now list three "mapping properties" of the little Fourier transformation. The first mapping property is that it is ***linear*** in the sense that

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \tag{4.3}$$

for all functions u, v and scalars α, β. This is a simple consequence of the formula (4.2).

The second mapping property is that the little Fourier transform is ***invertible***, meaning that there is a transform f^{-1} that takes in sequences in $l^2(\mathbb{Z})$ and gives out functions in $\mathcal{L}^2([-L, L])$ in such a way that

$$f^{-1}(f(u)) = u \text{ for all functions } u \text{ in } \mathcal{L}^2([-L, L]) \tag{4.4}$$

LFT-first-inversio

and

$$f(f^{-1}(c)) = c \text{ for all sequences } c \text{ in } l^2(\mathbb{Z}). \tag{4.5}$$

LFT-second-inversi

The formula for f^{-1} is given by

$$f^{-1}(c) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}. \tag{4.6}$$

LFT-inverse

The first inversion property (4.4) follows from the fact that the Fourier series of a function converges to the function (except at jump points, which we can ignore). If we apply (4.6) to $f(u)$ we get

$$f^{-1}(f(u)) = \sum_{k=-\infty}^{\infty} f(u)_k e^{i\frac{k\pi}{L}x},$$

which we know converges to $u(x)$. The second inversion property (4.5) is discussed in more detail in §**??**.

• section reference needed

The third mapping property is that the little Fourier transform is an ***isomorphism***. This means that there is a direct relationship between the norm of a function u and the norm of its transform $f(u)$. In particular, we have

$$\|f(u)\|^2 = \frac{1}{2L}\|u\|^2. \qquad (4.7)$$

LFT-isomorphism

This means that (up to an overall multiplicative factor) the little Fourier transform of a function is the "same size" as the function itself. The proof of this fact is discussed in §**??**.

• section reference needed
• needed

Exercise 4.2.1. *take previous FS examples and look at different parts of the series. What do low k terms do? What do high k terms do?*

Use a modification of the following code.

```
var('k')
start=20
stop =30

f(x) = (4/pi)* sum(sin((2*k+1)*x)/(2*k+1),k,start,stop)

plot( f(x), (x,0,4*pi), figsize = [4,3])
```

Exercise 4.2.2. *Set $L = 1$ and consider function*

$$u(x) = \begin{cases} (\#)(a - |x|) & \text{if } |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

1. *Find the number $(\#)$ so that $\|u\| = 1$.*

2. *Compute $f(u)$*

3. *Make a plot of $f(u)_k$ for large/small values of a.*

4. *Now set $a = 1/2$. Make a plot of the partial sum*

$$\sum_{|k|\le 5} f(u)_k e^{ik\pi x}.$$

How closely does this describe u?

5. *Still with $a = 1/2$. Make a plot of the partial sum*

$$\sum_{5\le|k|\le 100} f(u)_k e^{ik\pi x}.$$

What parts of the function u does this describe?

4.3 Properties of little Fourier transform

We now establish two more properties of the little Fourier transform. We won't use these properties directly in this course. Rather, we are demonstrating the properties for the little Fourier transform in preparation for our upcoming discussion of the "big" Fourier transform.

The first property is the "multiplication property" of the little Fourier transform. Suppose we have two functions u, v in $\mathcal{L}^2([-L, L])$; consider the two sequences $f(u)$ and $f(v)$. We can multiply these two lists of numbers in a very simply way: just multiply $f(u)_k f(v)_k$ for each k. This raises an interesting question? What operation on the original functions u and v does this multiplication of $f(u)_k f(v)_k$ correspond to?

To answer this question, we use the formula (4.2) which implies, provided we periodically extend u, v, that

$$\begin{aligned} f(u)_k f(v)_k &= \frac{1}{(2L)^2} \int_{-L}^{L} \left[\int_{-L}^{L} u(x)v(y)e^{-i\frac{k\pi}{L}(x+y)}\, dx \right] dy \\ &= \frac{1}{2L} \int_{-L}^{L} \left[\frac{1}{2L} \int_{-L}^{L} u(z)v(x-z)\, dz \right] e^{i\frac{k\pi}{L}x}\, dx \end{aligned}$$

Notice that this is simply the little Fourier transform of the stuff in square brackets, which is a function of x.

The quantity in square brackets above is called the ***convolution product*** of the functions u and v, and is given the symbol $u * v$. Thus the convolution product is defined by the formula

$$(u * v)(x) = \frac{1}{2L} \int_{-L}^{L} u(z)v(x-z)\, dz.$$

The convolution product is a way to multiply two functions together and obtain another function, but it is very different from the "usual" way of multiplying functions. (The "usual" way of multiplying functions is also called "pointwise multiplication.") The ***multiplication property*** of the little Fourier transform is that the convolution product of two functions is taken to the pointwise product of the transforms, and is expressed in the following formula:

$$f(u * v)_k = f(u)_k f(v)_k.$$

The convolution product of two functions is perhaps a little bit difficult to interpret. Before we attempt to do so, let's consider two examples.

Example 4.3.1

Fix two positive integers $p \neq q$ and let u, v be the functions given by

$$u(x) = \cos\left(\frac{p\pi}{L}x\right) \quad v(x) = \cos\left(\frac{q\pi}{L}x\right)$$

We compute the convolution

$$\begin{aligned}(u * v)(x) &= \frac{1}{2L}\int_{-L}^{L} u(z)v(x-z)\,dz \\ &= \frac{1}{2L}\int_{-L}^{L} \cos\left(\frac{p\pi}{L}z\right)\cos\left(\frac{q\pi}{L}(x-z)\right)\,dz \\ &= 0\end{aligned}$$

Notice that $f(u)_k = 0$ unless $k = \pm p$ and $f(v)_k = 0$ unless $k = \pm q$. Thus it is easy to see from the multiplication property of little Fourier transform that we must have $f(u)_k f(v)_k = 0$ for each k. This is consistent with our calculation that $u * v = 0$.

We interpret this example to mean that the convolution of two functions with totally different length (or frequency) scales is zero.

Example 4.3.2

For each local x_0 in the interval $[-L, L]$ and each small number a let u_{x_0} be the pulse function of size A concentrated at x_0, given by

$$u_{x_0}(x) = \begin{cases} 0 & \text{if } |x - x_0| > a \\ A\frac{L}{a} & \text{if } |x - x_0| \leq a. \end{cases}$$

Let v be any function in $\mathcal{L}^2([-L, L])$. We compute

$$(u_{x_0} * v)(x) = \frac{A}{2a}\int_{(x-x_0)-a}^{(x-x_0)+a} v(y)\,dy.$$

If we take a to be very small, then we have

$$(u_{x_0} * v)(x) \approx Av(x - x_0).$$

Thus if we take a function concentrated near x_0 and convolve it with v, the result is approximately the function v, but shifted by x_0.

If we think of a generic function u as being approximately built by the sum of pulses (of various strengths) at each location, then we can interpret the convolution product $u*v$ as giving us the sum of the various shifts of v, with each shift having the strength of the corresponding pulse.

The image that emerges from the two examples above is the following:

- On one hand, the convolution product can be seen as a measure of the extent to which the various frequencies (or length scales) of the two functions overlap. This is made clear by the multiplication property formula.
- On the other hand, the convolution product can be understood as a "shift to each point and scale by the size of the pulse at that point" which I think of as a "maximally dispersed multiplication."

The second property of the little Fourier transform that we present in this section is the ***derivative property***, which states that

$$f(u')_k = i\left(\frac{k\pi}{L}\right) f(u)_k. \tag{4.8}$$ LFT-derivative

We can interpret this formula to mean that the process of taking a derivative of a function corresponds to multiplying each of the frequency amplitudes by a number that is small for large scale (low frequency) structures and large for small scale (high frequency) structures. This makes sense because small scale structures of functions have, by definition, fluctuations on shorter length scales and thus contribute more to the derivative of a function.

Exercise 4.3.1. *Let u and v be functions in $\mathcal{L}^2([-L, L])$ that are periodically extended to functions on the entire real line. Show that $u*v = v*u$, meaning that the convolution product is commutative.*

Exercise 4.3.2 (Relies on Exercise 3.5.1)**.** *Compute the convolution of square wave and triangle wave.*

Exercise 4.3.3. *Verify that (4.8) holds. [Hint: use (4.2).]*