Chapter 2

The one dimensional wave equation

2.1 Derivation of the wave equation

sec:derive-1D-wave

In the previous chapter we saw how the principle of conservation of energy leads to the simple harmonic oscillator equation. We now apply the same sort of logic to a more complicated problem: the oscillation of a string. The first thing we need to do is figure out how to describe the motion of a string using a function. We can accomplish this by supposing that the string is confined to move in the xy plane, and that the string lies along the x axis when it is at rest. The motion of the string can be described by the displacement in the y direction. We let u measure this displacement. Since the displacement depends both on location x along the string that was displaced and on the time t at which the displacement occured, the displacement u is a function of t and x; we write $u = u(t, x)$.

We need to make some assumptions about what happens at the ends of the string. There are many options for how to handle the ends of the string. For the time being, we assume that we are dealing with one of the following situations:

Local displacements of an infinite string If we are focused on the oscillations of a very long string, we might assume that the displacement is localized to some region of the string. Mathematically this means that u is defined for all

$-\infty < x < \infty$ and that we are assuming both

$$u(t,x) \to 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,x) \to 0 \qquad \text{as} \quad x \to \pm\infty. \tag{2.1}$$

GeneralBC:infinite-stri

Periodic displacements of an infinite string If we are focused on the oscillations of a very long string we might alternatively assume that the displacement is spatially periodic with period $2L$. Mathematically we can express this by having u defined for $-L \leq x \leq L$ and by assuming that

$$u(t,-L) = u(t,L) \quad \text{and} \quad \frac{\partial u}{\partial x}(t,-L) = \frac{\partial u}{\partial x}(t,L). \tag{2.2}$$

GeneralBC:periodic

The condition (2.2) is a called the ***periodic boundary conditions***.

Finite string with fixed endpoints Another physically interesting situation is where the string has finite length L and the endpoints of the string are fixed. Mathematically we express this by having u defined for $0 \leq x \leq L$ and assuming

$$u(t,0) = 0 \quad \text{and} \quad u(t,L) = 0. \tag{2.3}$$

GeneralBC:Dirichlet

The condition (2.3) is called the ***Dirichlet boundary condition***.

Finite string with reflective endpoints Finally, we might assume that the string has finite length L and that the endpoints are "reflective" in the sense that the slope of the displacement is zero at endpoints. Mathematically we express this by having u defined for $0 \leq x \leq L$ and assuming

$$\frac{\partial u}{\partial x}(t,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,L) = 0. \tag{2.4}$$

GeneralBC:Neumann

The condition (2.4) is called the ***Neumann boundary condition***.

Since the conditions above are imposed at the endpoints of the spatial domain, these conditions are usually called ***boundary conditions***.

Assuming that the string satisfies one of the four boundary conditions above, we now define some sort of "energy" for the string. First, we focus on kinetic energy. Let's assume that the string has a linear mass density that is constant in x, even as the string moves and stretches. (Think about how reasonable this is... it's actually better than it might first appear!) Call the linear mass density ρ. The little piece of string at location x has mass $\rho\, dx$ and vertical velocity $\frac{\partial u}{\partial t}$. It makes sense that the "kinetic energy" at that point would be $\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 dx$ and that the total kinetic energy of the string would be

$$K = \frac{1}{2}\rho \int_{\text{string}} \left(\frac{\partial u}{\partial t}\right)^2 dx.$$

Note that ρ is the "linear-densitized" version of the constant m appearing in the simple harmonic oscillator.

Let's now think about "potential energy," which is supposed to measure the "energy" associated with the location of the string. That is, we just saw a picture of a string at some time, we should be able to compute the potential energy without knowing anything about the velocity at that time. Another way to think about potential energy is this: Suppose that the string is being held in some physical configuration. To what extent would the fact that the string is in that configuration lead to physical motion (and, perhaps, kinetic energy) if it was suddenly released?

If the string started out lying on the x axis, we would not expect any motion to suddenly emerge, so we would like that configuration to have zero potential energy. Likewise, if the string–endpoints and all–was just shifted up by some fixed amount, then we again would not expect any motion. Thus we want to assign zero potential energy to sections of the string which are described by u being constant in x.

On the other hand, if initially there is a part of the string is bent steeply, then we expect there to be quite a bit of motion when the string is released. This motivates us to define the potential energy to be

$$V = \frac{1}{2}k \int_{\text{string}} \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Notice that in order for K and V to have the same units, it must be that k has units of (mass)/(time)^2(length), which is a linear-densitized version of the corresponding constant for the simple harmonic oscillator.

The total energy $E = K + V$ is thus

$$E = \rho\frac{1}{2} \int_{\text{string}} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + c^2 \left(\frac{\partial u}{\partial x}\right)^2 \right\} dx,$$

where the constant $c = \sqrt{k/\rho}$ has units of (length)/(time); below we see how to interpret c as a velocity.

We now impose the condition that E be constant in time, which requires that

$$0 = \rho \int_{\text{string}} \left\{ \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x} \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial t} \right] \right\} dx. \tag{2.5}$$ PreDeriveWave-1

Integrating by parts we see that

$$\int_{\text{string}} \frac{\partial u}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) dx = \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right]_{\text{endpoints}} - \int_{\text{string}} \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial t} dx. \tag{2.6}$$ PreWave-IBP

Notice that any of the four conditions (2.1)–(2.2)–(2.3)–(2.4) imply that the "endpoints" term vanishes; see Exercise 2.1.1. Thus (2.5) becomes

$$0 = \rho \int_{\text{string}} \frac{\partial u}{\partial t} \left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \right) dx.$$

Since we do not want to restrict the velocity $\frac{\partial u}{\partial t}$, we conclude that in order to have energy conserved we must have

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.7}$$ PreWave

at each time t and location x.

Equation (2.7) is called the ***(one-dimensional) wave equation***. Solutions to the wave equation must also satisfy the one of the conditions (2.1)–(2.2)–(2.3)–(2.4) in order to describe the amplitude of an oscillating string.

rify-boundary-term-vanishes **Exercise 2.1.1.** *For each of the four possible endpoint assumptions, carefully explain why the "endpoints" term in (2.6) vanishes.*

2.2 Putting the wave in the wave equation

In the previous section we gave a derivation of the wave equation. In this section we want to gain some understanding of the ways in which the equation does, in fact, describe phenomena we regard as being "wave-like." This is a bit more difficult that it might first seem because there are many phenomena which might be described as "wave-like." (For a more thorough introduction to wave-like equations, see Roger Knobel's excellent book *An introduction to the Mathematical Theory of Waves*, published by the AMS.)

One way to characterize waves is to require that "disturbances or signals propagate at some finite speed." Suppose $\phi\colon \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function such that $\phi(z) \to 0$ and $\phi'(z) \to 0$ as $z \to \infty$. Then (see Exercise 2.2.1) the function $u(t,x) = \phi(x - ct)$ is a solution to the wave equation (2.7) satisfying condition (2.1). Such a solution to the wave equation is called a ***traveling wave*** because it is comprised of a fixed displacement profile (given by ϕ) traveling along the x axis at velocity c.

• **To do: put in Desmos links as well**

Example 2.2.1

Set $c = 1$ and consider the function $\phi(z) = e^{-z^2}$. The corresponding traveling wave $u(t,x) = \phi(x - t)$ can be animated in Sage by the following code:

```
phi(x) = exp(-x^2)
a = animate([phi(x-t) for t in srange(-6,6,0.5)], xmin
    =-5, xmax=5, ymax=1.2, ymin=-.1, figsize=[4,2])
a.show()
```

Alternatively, see this link.

Not all solutions to (2.7) are traveling waves. For example, let ω be any positive constant. Then the function

$$u(t,x) = \cos(\omega t)\cos(\frac{\omega}{c}x) \tag{2.8}$$

FirstStandingWave

is a solution to (2.7). The function in (2.8) is called a ***standing wave*** because it takes the form

$$u(t,x) = A(t)\psi(x), \tag{2.9}$$

which we interpret as being comprised of a spatial shape $\psi(x)$ that is scaled by amplitude function $A(t)$ as time evolves.

Example 2.2.2

Set $c = 1$ and $\omega = 1$. The standing wave (2.8) can be animated in Sage with the following code:

```
a = animate([cos(x)*cos(t) for t in srange(0,2*pi,0.3)],
    xmin=-3*pi, xmax=3*pi, ymax=1.2, ymin = -1.2, figsize
    =[4,2])
a.show()
```

Alternatively, see this link.

HW:verify-traveling-wave

Exercise 2.2.1. *Suppose $\phi\colon \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function such that $\phi(z) \to 0$ and $\phi'(z) \to 0$ as $z \to \infty$. Show that the functions*

$$u_{\text{left}}(t,x) = \phi(x+ct)$$
$$u_{\text{right}}(t,x) = \phi(x-ct)$$

are solutions to (2.7). Describe how these functions behave in time.

Ex:wave-equation-is-linear

Exercise 2.2.2.

1. *Show that the wave equation (2.7) is linear. That is, suppose both u_1 and u_2 are solutions. Show that $\alpha u_1 + \beta u_2$ is also a solution for any constants α, β.*

2. *Let's illustrate the linearity of (2.7) with an example. With $c = 1$ explain why both*

$$u_1(t,x) = \frac{2}{1+(x+t)^4}$$
$$u_2(t,x) = \frac{1}{\sqrt{1+(x-t)^2}}$$

   are solutions. Then explain why $u(t,x) = u_1(t,x) + u_2(t,x)$ is also a solution. Have Sage animate this solution for you. Explain what you observe.

3. *One can physically interpret the superposition principle as indicating that "individual waves do not interact." Do "real world waves" that behave this way? Discuss several different types of examples (water waves, sound waves, light, etc.).*

Ex:TowardsStandingWaves

Exercise 2.2.3. *This exercise connects the ideas of standing waves and traveling waves.*

For simplicity, we set $c = 1$. Consider the solution

$$u(t,x) = \cos(x-t) + \alpha\cos(x+t),$$

where α is some parameter.

1. *First, show that when $\alpha = 0$ the solution is simply a traveling cosine wave. Use the following Sage code to animate the solution:*

```
var('x')
a=0.0
waveplot = animate([cos(x-t) + a*cos(x+t) for t in srange
    (0,6,0.5)],xmin=-2*pi, xmax=2*pi, ymax=2, ymin = -2,
    figsize=[4,2])
waveplot.show()
```

2. *Now consider the case when* $\alpha = 0.1$. *Explain how to interpret the the solution as the sum of two traveling waves, and explain the visual effect of the second wave.*

3. *Repeat for* $\alpha = 0.2, 0.3, 0.4, \ldots, 0.9$. *What happens as you slowly increase* α*?*

4. *Now set* $\alpha = 1$. *What is the behavior of the solution?*

5. *Use the trigonometric identity for the sum of cosines to show that when* $\alpha = 1$ *the solution can be written*

$$u(t,x) = 2\cos(t)\cos(x). \tag{2.10}$$ PreStandingWave

Explain how to interpret this as a standing wave.

6. *What can you conclude from this experiment?*

2.3 A systematic look at standing waves

sec:standing-waves

Mathematically, standing wave solutions to (2.7) are somewhat analogous to eigensolutions of linear ODEs: both take the form

$$(\text{function of } t)(\text{spatial object}).$$

In the ODE case, the "spatial object" is a vector. In the case of a standing wave, the spatial object is a function of the spatial variable x. In Part 2 of this course, we make the analogy between vectors and functions of spatial variables much more concrete.

For ODEs we found that we were able to construct complete, independent collections of eigensolutions. Our plan for studying the wave equation is to construct complete, independent collections of standing wave solutions. To do this, we need to make a more systematic analysis of standing wave solutions to (2.7).

For simplicity, we set the constant c equal to 1, so that the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{2.11}$$

1D-wave

Standing wave solutions take the form

$$u(t,x) = A(t)\psi(x). \tag{2.12}$$

1D-standing-ansatz

Inserting (2.12) in to (2.11) we find that

$$\underbrace{\frac{1}{A}\frac{d^2 A}{dt^2}}_{\clubsuit} = \underbrace{\frac{1}{\psi}\frac{d^2\psi}{dx^2}}_{\heartsuit}.$$

We now make an important observation: the quantity $\clubsuit$ is a function of t only, while the quantity $\heartsuit$ is a function of x only. Thus

$$\frac{d}{dt}[\clubsuit] = \frac{d}{dt}[\heartsuit] = 0,$$

which means that $\clubsuit$ is a constant and, since $\clubsuit = \heartsuit$, that $\heartsuit$ is equal to that same constant. We call the constant λ. Therefore we can construct standing wave solutions to (2.11) of the form (2.12) by solving the ODEs

$$\frac{d^2 A}{dt^2} = \lambda A \quad \text{and} \quad \frac{d^2\psi}{dx^2} = \lambda\psi \tag{2.13}$$

1D-first-separatio

for some constant λ.

This is good news, because we already know how to solve ODEs like the ones in (2.13). In fact, we can solve them for any value of λ. For example, suppose $\lambda = 4$. Then we can take $A(t) = e^{-2t}$ and $\psi(x) = e^{2x}$. The resulting solution is

$$u(t,x) = e^{-2t}e^{2x} = e^{2(x-t)}, \tag{2.14}$$

1D-exp-traveling-w

which we can interpret as a right-moving traveling wave solution based on the shape $\phi(z) = e^{2z}$.

However, the solution (2.14) does not satisfy any of the boundary conditions discussed in §2.1. Thus if we want our standing wave solutions to describe one of those situations, we must find a way to incorporate those conditions in to the differential equations (2.13). Since we want the boundary conditions to hold at all times t, the easiest thing to do is to require that ψ satisfy the corresponding boundary condition. In particular,

- For modeling local displacements of an infinite string, the function ψ is defined for $-\infty < x < \infty$ and the boundary condition (2.1) becomes

$$\psi(x) \to 0 \quad \text{and} \quad \psi'(x) \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{2.15}$$

BC:infinite-string

- For modeling periodic displacements of an infinite string, the function ψ is defined for $-L \le x \le L$ and the boundary condition (2.2) becomes

$$\psi(-L) = \psi(L) \quad \text{and} \quad \psi'(-L) = \psi'(L). \tag{2.16}$$

BC:periodic

  We call (2.16) the ***periodic boundary condition***.

- For modeling a finite string with fixed endpoints, the function ψ is defined for $0 \le x \le L$ and the boundary condition (2.3) becomes

$$\psi(0) = 0 \quad \text{and} \quad \psi(L) = 0. \tag{2.17}$$

BC:Dirichlet

  We call (2.17) the ***Dirichlet boundary condition***.

- For modeling a finite string with reflective endpoints, the function ψ is defined for $0 \le x \le L$ and the boundary condition (2.4) becomes

$$\psi'(0) = 0 \quad \text{and} \quad \psi'(L) = 0. \tag{2.18}$$

BC:Neumann

  We call (2.18) the ***Neumann boundary condition***.

In order to keep our discussion simple, let's focus on the case of the Dirichlet boundary condition. So we want to find functions A and ψ that satisfy (2.13) and (2.17) for some constant λ. Since the boundary condition is a condition on ψ, it makes sense to first solve

$$\frac{d^2\psi}{dx^2} = \lambda\psi, \qquad \psi(0) = 0, \quad \psi(L) = 0, \tag{2.19}$$

1D:Dirichlet-bvp

and then solve for the function A.

The combination of the differential equation and boundary condition in (2.19) is called a ***boundary value problem***, given the acronym BVP. Our perspective is to view (2.19) as an ***eigenvalue problem***, analogous to the eigenvalue problems we encountered in the ODEs course. Most abstractly, an eigenvalue problem is an equation of the form

$$[\text{operation}](\text{object}) = (\text{number})(\text{object}). \tag{2.20}$$

In the ODEs course, the "object" was a vector and the "operation" was multiplication by a matrix. For the wave equation, the "object" is the function ψ and the "operation" is the process of taking two derivatives. Due to this analogy we refer to possible values of λ as eigenvalues. (In fact, this is why we chose the letter λ!) We refer to the functions ψ that satisfy (2.19) as ***eigenfunctions***.

The boundary condition in the BVP restrict which eigenvalues λ are possible. To see this, we multiply the differential equation in (2.19) by ψ and integrate to obtain

$$\begin{aligned}\int_0^L \psi(x)\lambda\psi(x)dx &= \int_0^L \psi(x)\frac{d^2\psi}{dx^2}(x)\,dx \\ &= \left[\psi(x)\frac{d\psi}{dx}(x)\right]_0^L - \int_0^L \left(\frac{d\psi}{dx}(x)\right)^2 dx,\end{aligned}$$

where in the second line we have used integration by parts. Notice that the Dirichlet boundary condition implies that the boundary term vanishes, and thus

$$\lambda \underbrace{\int_0^L (\psi(x))^2\,dx}_{\geq 0} = -\underbrace{\int_0^L \left(\frac{d\psi}{dx}(x)\right)^2 dx}_{\geq 0}. \tag{2.21}$$

1D-eigenvalue-has-

Since both of the integrals are non-negative, it must be the case that $\lambda \leq 0$.

The identity (2.21) also shows that if $\lambda = 0$ then $\frac{d\psi}{dx} = 0$, which implies that ψ is constant. However, the only constant function satisfying the Dirichlet boundary condition is the zero function. Thus in order to get nonzero standing wave solutions we must have $\lambda < 0$.

Since $\lambda < 0$ we can write $\lambda = -\omega^2$ for some positive constant ω. Making this substitution in (2.19) yields

$$\frac{d^2\psi}{dx^2} = -\omega^2\psi, \qquad \psi(0) = 0 \quad \psi(L) = 0.$$

From our differential equations course we know that solutions to this equation are linear combinations of

$$\cos(\omega x) \quad \text{and} \quad \sin(\omega x). \tag{2.22}$$

1D-basic-shapes

Since $\cos(0) = 1$, the function $\cos(\omega x)$ cannot the boundary condition at $x = 0$. Thus we rule out this solution.

The function $\sin(\omega x)$ automatically satisfies the boundary condition at $x = 0$. In order to satisfy the boundary condition at $x = L$ we must have

$\sin(\omega L) = 0$. This means that ωL must be a multiple of π. This gives us an infinite list of possible values for ω:

$$\omega_1 = \frac{\pi}{L}, \quad \omega_2 = \frac{2\pi}{L}, \quad \dots, \quad \omega_k = \frac{k\pi}{L}, \quad \dots$$

and corresponding infinite list of possible values for $\lambda_1, \lambda_2, \dots$ given by the formula

$$\lambda_k = -\left(\frac{k\pi}{L}\right).$$

For each λ_k there is a solution ψ_k to the boundary value problem (2.19):

$$\psi_k(x) = \sin(\omega_k x) = \sin\left(\frac{k\pi}{L}x\right). \tag{2.23}$$ 1D:Dirichlet-eigenfunctions

Finally, in order to construct standing wave solutions, we need to find the corresponding functions $A(t)$. For each $k = 1, 2, 3, \dots$ we need a function A_k satisfying

$$\frac{d^2 A_k}{dt^2} = -\omega_k^2 A_k. \tag{2.24}$$ 1D:coefficient-ode

From the ODEs course, we know that the general solution to (2.24) is

$$A_k = a_k \cos(\omega_k t) + b_k \sin(\omega_k t),$$

where a_k and b_k are constants. The result is that we end up with an infinite list of standing wave solutions. For $k = 1, 2, 3, \dots$ we have the solution

$$u_k(t, x) = a_k \cos(\omega_k t)\sin(\omega_k x) + b_k \sin(\omega_k t)\sin(\omega_k x), \qquad \text{where} \qquad \omega_k = \frac{k\pi}{L}.$$

Alternatively, we see that for any $k = 1, 2, 3 \dots$ the functions

$$\cos\left(\frac{k\pi}{L}t\right)\sin\left(\frac{k\pi}{L}x\right) \qquad \text{and} \qquad \sin\left(\frac{k\pi}{L}t\right)\sin\left(\frac{k\pi}{L}x\right) \tag{2.25}$$ Dirichlet-eigensolutions

are each solutions.

We have succeeded in constructing a list of all possible standing wave solutions to the wave equation that also satisfy the Dirichlet boundary condition. We refer to these solutions as ***eigensolutions*** because they are constructed from the eigenfunctions ψ_k that satisfy the BVP (2.19).

Before we close this section, we make two remarks about boundary conditions

- Notice that imposing the boundary condition forced the eigenvalues λ to all be negative. We demonstrated this using integration-by-parts. The fact that the eigenvalues are negative is what led to the standing wave solutions to be oscillatory: if $\lambda > 0$, then we could not get cosine or sine functions for $A(t)$. This is a more general phenomenon that will show up again.
- Notice also that imposing boundary conditions led to there being a discrete list of possible eigenvalues. (Before we imposed the boundary conditions, any value of λ was admissible!) Thus there is a certain sense in which boundary conditions lead to the discretization of possible solutions. In physics, this phenomenon is related to the concept of "quantization." In mathematics, such a countable list of eigenvalues is called a "discrete spectrum."

c-eigenfunctions-first-pass **Exercise 2.3.1.**

1. *Suppose that ψ is a function defined on $[-L, L]$ satisfying*

$$\frac{d^2\psi}{dx^2} = \lambda\psi \tag{2.26}$$

1D-periodic-psi-ec

and the periodic boundary condition (2.16). Use integration by parts to show that $\lambda = -\omega^2$ for some real number ω.

2. *Find the infinite list of all solutions to (2.26) satisfying the periodic boundary conditions (2.16). (You should, in fact, get two infinite lists – one involving cosines and one involving sines.)*
3. *Find all standing wave solutions to (2.11) that satisfy the periodic boundary conditions (2.2).*
4. *Suppose now that we are willing to work with complex numbers in our solutions to the wave equation. Show that the list of complex solutions to (2.26)–(2.16) is*

$$\psi_k(x) = e^{i\frac{k\pi}{L}x}, \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

What are the corresponding eigensolutions to the wave equation?

Exercise 2.3.2. *Consider the Dirichlet standing wave solutions in (2.25).*

1. *Since these are solutions to the wave equation, they should have constant energy. Compute the energy of the k^{th} solution. Express the energy in terms of the frequency ω_k at which the standing wave oscillates.*

2. *What is the relationship between the (relative) size of the frequency of oscillation and the size of the energy? Express your answer in the form As the frequency ω grows larger, the energy tends to ____ . Etc.*

3. *What is the relationship between the frequency of the oscillation and the spatial scale of the solution? Express your answer in the form When the frequency ω of the standing wave is large, the spatial scale of the wave tends to ____ . Etc.*

4. *Summarize the connections/relationships between energy, length scale, and frequency of oscillation.*

Exercise 2.3.3. *In Exercise 2.2.2 you showed that the wave equation is linear, and thus that we can use the superposition principle to generate new solutions from old ones. In this exercise you get to explore this in the case that $L = \pi$ and Dirichlet boundary conditions are enforced.*

1. *Both*

$$\cos(t)\sin(x) \quad \text{and} \quad \cos(4t)\sin(4x)$$

are solutions. Spend some time playing around with different linear combinations of these solutions. You might it helpful to make use of the following: https://www.desmos.com/calculator/ndrvima2l2

2. *Check out this solution: https://www.desmos.com/calculator/yppajxal4f What's happening here?*

2.4 The initial boundary value problem

section:1D-IBVP

• rewrite?

In the differential equations course, we saw that that the eigensolution approach to studying linear differential equations involved three parts:

- **Construct a independent collection of eigensolutions.** By "independent" we mean that no one of the eigensolutions can be expressed as a linear combination of the others.

- **Show that collection of eigensolutions is complete,** meaning that any solution can be constructed as a linear combination of eigensolutions. This relies on the superposition principle.

- **Understand the behavior of general solutions by understanding the behavior of the eigensolutions.**

In the previous section we constructed a list of eigensolutions to the wave equation with Dirichlet boundary conditions, the standing wave solutions in (2.25). Thus we are left with the following questions:

- Is the collection of eigensolutions (2.25) complete? Are these functions independent from one another?
- Can all solutions to the wave equation (satisfying the Dirichlet boundary condition) be constructed as a linear combination of these eigensolutions?
- What do these eigensolutions tell us about the behavior of general solutions of the wave equation (satisfying the Dirichlet boundary condition)?

In order to address these questions we introduce the concept of the "initial boundary value problem." This concept is motivated by the initial value problem studied in the ODE course.

Recall that a second-order ODE of the form[1]

$$a\frac{d^2u}{dt^2} + b\frac{du}{dt} + cu = f$$

requires two initial conditions, the initial value $u_0 = u(0)$ and the initial velocity $v_0 = u'(0)$. The fundamental theorem of ODEs implies that once these initial values are specified, there exists exactly one corresponding solution to the ODE.

Since the wave equation involves two time derivatives, it is reasonable to expect that one should be able to similarly specify an initial value and an initial velocity. However, if $u(t,x)$ is a solution to the wave equation, the value at $t = 0$ is a function $u_0(x) = u(0,x)$. Similarly, the initial velocity for the wave equation is a function $v_0(x) = \frac{\partial u}{\partial t}(0,x)$. The wave equation also has the additional complication that we must satisfy one of the boundary conditions described in §2.1. Thus for the wave equation we can pose the following ***initial boundary value problem***:

> Consider one of the boundary conditions (2.1), (2.2), (2.3), or (2.4) and corresponding spatial domain Ω. Suppose u_0 and v_0 are functions defined on the spatial domain Ω. A function $u = u(t,x)$, defined for $t \in [0,T]$ and $x \in \Omega$ is a solution to the corresponding initial boundary value problem (IBVP) if
>
> - u satisfies the given boundary condition,

[1] Assume, for simplicity, that a, b, c are differentiable functions with $a(0) \neq 0$.

- u satisfies the wave equation (2.11), and
- u satisfies the initial conditions

$$u(0,x) = u_0(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(0,x) = v_0(x). \tag{2.27}$$ 1D-IC

Remark 2.4.1. *Let $\mathring{\Omega}$ denote the interior of the spatial domain Ω. Technically, we only require that the wave equation (2.11) hold for $t \in (0,T)$ and $x \in \mathring{\Omega}$, only require that the initial condition (2.27) hold for $x \in \mathring{\Omega}$, and only require that the boundary condition hold for $t > 0$.*

Let's focus our attention on the Dirichlet initial boundary value problem, by which we mean the IBVP with Dirichlet boundary condition (2.3) on domain $\Omega = [0, L]$. Our plan for constructing solutions to the Dirichlet IBVP is to consider linear combinations of the eigensolutions (2.25). Thus we fix functions u_0 and v_0 and seek a solution to the IBVP of the form

$$u(t,x) = \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}t\right) \sin\left(\frac{k\pi}{L}x\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}t\right) \sin\left(\frac{k\pi}{L}x\right), \tag{2.28}$$ 1D-ansatz

where a_k and b_k are some unknown constants. By construction, functions of this form satisfy the wave equation and satisfy the Dirichlet boundary condition. Thus we have two questions to address:

- Can we choose the constants a_k, b_k so that the initial conditions are satisfied?
- Do we need to worry about these infinite sums? For example, do they converge?

If we evaluate (2.28) at $t = 0$ we see that in order for the first initial condition to be satisfied we must have

$$u_0(x) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi}{L}x\right).$$

Similarly, if we apply ∂_t to (2.28) and then evaluate at $t = 0$ we see that in order for the second initial condition to be satisfied we need

$$v_0(x) = \sum_{k=1}^{\infty} b_k \frac{k\pi}{L} \sin\left(\frac{k\pi}{L}x\right).$$

Thus we see that our plan for obtaining solutions to the Dirichlet IBVP of the form (2.28) rests on begin able to answer the following question:

> Given a function f defined on $[0, L]$, can we choose constants a_k so that the functions f_n, defined by
>
> $$f_n(x) = \sum_{k=1}^{n} a_k \psi_l(x)$$
>
> converge to f as $n \to \infty$?

This question was addressed in 1822 by mathematician Jean-Baptiste Joseph Fourier, who gave an affirmative answer in his book *Théorie analytique de la chaleur* (Analytic theory of heat). (It would, however, take the development of more mathematics theory in order to put Fourier's work on solid mathematical footing). In the next chapter we develop the theory needed to understand and use Fourier's work. Once we have developed these tools, we return to the initial boundary value problem for the one-dimensional wave equation.

• **exercise needed**

Exercise 2.4.1.

Exercise 2.4.2. *In this exercise you address the uniqueness of solutions to the Dirichlet IBVP.*

1. *Suppose u is a function of time t and spatial location $x \in [0, L]$. Motivated by the discussion in §2.1, we define the energy at time t of u by*

   $$E[u](t) = \frac{1}{2} \int_0^L \left\{ \left(\frac{\partial u}{\partial t}(t, x) \right)^2 + \left(\frac{\partial u}{\partial x}(t, x) \right)^2 \right\} dx.$$

   Suppose that u is a twice differentiable function such that at each time t it satisfies the Dirichlet boundary condition. Explain why if $E[u](t) = 0$ then $u(t, x) = 0$ for each $x \in [0, L]$.

2. *Suppose that u is a twice differentiable function such that at each time t it satisfies the wave equation and the Dirichlet boundary condition. Show that*

   $$\frac{d}{dt} E[u](t) = 0.$$

3. *Suppose that u is a twice differentiable function such that at each time t it satisfies the wave equation, the Dirichlet boundary condition, and has initial conditions $u_0 = 0$, $v_0 = 0$. Show that $E[u](t) = 0$ for all $t > 0$. Explain why this implies that $u = 0$.*

4. *Suppose now that* $\breve{u}$ *and* $\widehat{u}$ *are two solutions to the Dirichlet initial boundary value problem with the same initial conditions. Show that the function* $u = \breve{u} - \widehat{u}$ *satisfies the Dirichlet initial boundary value problem with initial conditions* $u_0 = 0$, $v_0 = 0$. *Explain how to conclude that* $\breve{u} = \widehat{u}$ *and therefore that solutions to the Dirichlet IBVP are unique.*