Hyperbolic Geometry and Laplace's Equation

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Agenda

My plan is to use the Yamabe problem to tell a story that connects hyperbolic geometry to "special cases" we study in ODEs course

I hope that you...

- gain a new perspective on some of your courses
- get a small sense of what I'm up to mathematically

Axiomatic approach to hyperbolic geometry

Neutral geometry

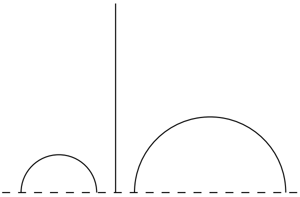
- Incidence axioms
- Order axioms
- Congruence axioms
- Continuity axioms

Hyperbolic parallel postulate

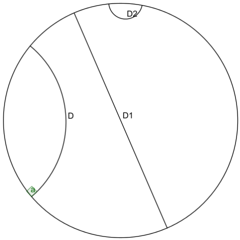
- For each line l and each point $A \notin l$ there exists two distinct lines that contain A and are parallel to l.

Models of the hyperbolic plane

Half-space model



Poincaré disk model



Credit: Jean-Christophe BENOIST

https://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model

Both of these models extend to higher dimensions

Scalar curvature

- Gauss-Bonnet formula for triangles

$$\mathrm{R}\,A + (\alpha + \beta + \gamma) = 2\pi$$

  Scalar curvature as angle deficit per unit area

$$\mathrm{R} = \frac{2\pi - \alpha - \beta - \gamma}{A} \tag{1}$$

- Subject to overall length scale
- Generalize to 3D using balls

$$vol(B_r) = vol_E(B_r)\left(1 - \frac{1}{30}\,\mathrm{R}\,r^2 + O(r^4)\right)$$

Riemannian structure

- Riemannian metric g: a dot product $\langle V, W \rangle_g$ at each point
- Length of paths

$$L = \int_{t_0}^{t_1} \sqrt{\langle X'(t), X'(t) \rangle_g} dt$$

- Lines are geodesics $\implies$ recover model for axioms

Scalar curvature II

- In coordinates $g = g_{ij} dx^1 dx^j$ and

$$\mathrm{R}[g] \sim g^{-1} g^{-1} \partial\partial g + g^{-1} g^{-1} g^{-1} (\partial g)(\partial g)$$

- length scale = multiply g by some constant

Half space model of hyperbolic space

$$\breve{g} = \frac{(dx^1)^2 + (dx^2)^2 + dy^2}{y^2}$$

Curvature $\mathrm{R}[\breve{g}] = -6$

Henceforth dimension $= 3$

Asymptotically hyperbolic manifolds

Definition

- Interior of compact manifold with boundary.
- Complete metric g
- Boundary geometry locally $\approx$ hyperbolic $\breve{g}$

Why care?

- Thurston: "most" geometry is hyperbolic
- General relativity
- Complete manifolds are interesting!

Yamabe's question

Given (M, g) can we rescale the metric to obtain constant scalar curvature?

$$g \mapsto \phi^4 g \qquad\qquad \phi > 0$$

Motivations

- 2D has uniformization. Is this the next best thing?
- Correct number of degrees of freedom.
- Conformal geometry.

Variational perspective

In the case of a compact manifold, Yamabe's problem has a variational formulation.

- Minimize

$$Q = \frac{\int_M \mathrm{R}[\phi^4 g]\, dV_{\phi^4 g}}{\left(\int_M 1\, dV_{\phi^4 g}\right)^{1/3}} = \frac{\int_M (8|\nabla\phi|_g^2 + \mathrm{R}[g]\phi^2) dV_g}{\left(\int_M \phi^6\, dV_g\right)^{1/3}} = \frac{E[\phi]}{\|\phi\|_{L^6(M)}^2}$$

- Nonlinear eigenvalue problem

$$-8\Delta_g \phi + \mathrm{R}[g]\phi = \lambda\phi^5 \tag{Y}$$

λ is the new scalar curvature

Yamabe problem in AH setting

In the AH setting we obtain

$$-8\Delta_g \phi + \mathrm{R}[g]\phi = -6\phi^5. \qquad \text{(Y-AH)}$$

together with the boundary condition $\phi \to 1$.

Main questions:

- existence
- uniqueness
- regularity

Existence and uniqueness

- 1992 Andersson-Chruściel-Friedrich;
- 2018 A-Isenberg-Lee-Stavrov

Existence: Fixed-point argument
Uniqueness: Ratio of two solutions satisfies a linear equation

Use linearization of

$$-8\Delta_g \phi + \mathrm{R}[g]\phi + 6\phi^5 = 0 \qquad \phi \to 1$$

Take $\phi = 1 + u$ to obtain

$$\Delta_g u - 3u = \underbrace{\frac{1}{8}(6 + \mathrm{R}[g])}_{f} + \underbrace{Q(u) + \frac{1}{8}(6 + \mathrm{R}[g])u}_{N(u)}.$$

Boundary regularity

Plan is to "integrate in y" the linearized form of the equation

$$\Delta_g u - 3u = f + N(u) \qquad u \to 0$$

- $f = f_1 y + f_2 y^2 + f_3 y^3 + \dots$
- If $u = O(y^n)$ then $N(u) = O(y^{n+1})$.
- Integrate one power at a time: $u = u_1 y + u_2 y^2 + u_3 y^3 + \dots$

Focus on the model problem $\Delta_{\breve{g}} u - 3u = f + N(u)$

Laplace operator in half-space model I

$$\breve{g} = \frac{(dx^1)^2 + (dx^2)^2 + dy^2}{y^2}$$

Dirichlet energy

$$E = \frac{1}{2} \int |\nabla u|^2_{\breve{g}} dV_{\breve{g}} = \frac{1}{2} \int y^2 (\text{grad}\, u) \cdot (\text{grad}\, u) y^{-3} dx^1 dx^2 dy$$

Using divergence theorem, the first variation is

$$\begin{aligned} \dot{E} &= \int y^{-1} (\text{grad}\, u) \cdot (\text{grad}\, \dot{u}) dx^1 dx^2 dy \\ &= - \int \underbrace{y^3 \,\text{div}(y^{-1}(\text{grad}\, u))}_{\Delta_{\breve{g}} u} \, \dot{u} \, \underbrace{y^{-3} dx^1 dx^2 dy}_{dV_{\breve{g}}} \end{aligned}$$

Linearized equation in half-space model

We computed

$$\begin{aligned}\Delta_{\breve{g}} u &= y^3 \operatorname{div}(y^{-1} \operatorname{grad} u) \\ &= y^3 \partial_y (y^{-2} y \partial_y u) + y^2 \left(\partial^2_{x^1} u + \partial^2_{x^2} u\right) \\ &= y\partial_y (y \partial_y u) - 2y\partial_y u + N(u)\end{aligned}$$

Linearized equation is

$$y\partial_y (y\partial_y u) - 2y\partial_y u - 3u = f + N(u)$$

Factored

$$(y\partial_y + 1)(y\partial_y - 3)u = f + N(u)$$

Pop quiz

Compute

$$y\partial_y(y^n \log y) = ???$$

Thus

$$(y\partial_y + 1)(y\partial_y - 3)(???) = \cdots + f_3 y^3 + \ldots$$

Smooth equations might not admit smooth solutions !

What's going on here?

Change variables $y = e^t$ so that $y\partial_y = \partial_t$.

$$(\partial_t - 3)(\partial_t + 1)u = f + N(u)$$

Homogeneous solutions: $e^{3t} = y^3$ and $e^{-t} = y^{-1}$.

Expand

$$\begin{aligned} f &= f_1 y + f_2 y^2 + f_3 y^3 + \dots \\ &= f_1 e^t + f_2 e^{2t} + \underbrace{f_3 e^{3t}} + \dots \end{aligned}$$

The f_3 term is a resonance!

From the differential equations course... $te^{3t} = y^3 \log y$

For the algebraist

Write as first order system

$$\partial_t \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{U} + \underbrace{\begin{pmatrix} f + N(u) \\ 0 \end{pmatrix}}_{F}$$

Eigenvalues of A are $-1, 3$

Duhamel's formula

$$U = \int_*^t \exp(A(t-s))F(s)ds$$

The algebraic structure of the equations forces log terms upon us.

Results with Isenberg-Lee-Stavrov

- Define Banach spaces capturing high interior regularity with limited boundary regularity

- Fredholm results for elliptic operators in this category

- Applications to Yamabe problem and Einstein constraint equations

Concluding remarks

- Yamabe problem connects parts of our curriculum that don't often talk to one another – this is fun!
- Hyperbolic geometry is interesting beyond axiomatic geometry
- All those special cases in ODEs are good for something
- Yay!

Thank you!