Hyperbolic Geometry and Laplace’s Equation

Paul T. Allen
Lewis & Clark College

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My plan is to use the Yamabe problem to tell a story that connects hyperbolic geometry to “special cases” we study in ODEs course.

I hope that you...

- gain a new perspective on some of your courses
- get a small sense of what I’m up to mathematically
Axiomatic approach to hyperbolic geometry

Neutral geometry
- Incidence axioms
- Order axioms
- Congruence axioms
- Continuity axioms

Hyperbolic parallel postulate
- For each line $l$ and each point $A \notin l$ there exists two distinct lines that contain $A$ and are parallel to $l$. 
Models of the hyperbolic plane

Half-space model

Poincaré disk model

Credit: Jean-Christophe BENOIST
https://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model

Both of these models extend to higher dimensions
Scalar curvature

- Gauss-Bonnet formula for triangles

\[ RA + (\alpha + \beta + \gamma) = 2\pi \]

Scalar curvature as angle deficit per unit area

\[ R = \frac{2\pi - \alpha - \beta - \gamma}{A} \] (1)

- Subject to overall length scale

- Generalize to 3D using balls

\[ \text{vol}(B_r) = \text{vol}_E(B_r) \left( 1 - \frac{1}{30} R r^2 + O(r^4) \right) \]
Riemannian structure

- Riemannian metric $g$: a dot product $\langle V, W \rangle_g$ at each point

- Length of paths

\[ L = \int_{t_0}^{t_1} \sqrt{\langle X'(t), X'(t) \rangle_g} \, dt \]

- Lines are geodesics $\implies$ recover model for axioms
Scalar curvature II

- In coordinates $g = g_{ij} dx^1 dx^j$ and
  \[ R[g] \sim g^{-1} g^{-1} \partial \partial g + g^{-1} g^{-1} g^{-1} (\partial g)(\partial g) \]

- length scale = multiply $g$ by some constant

Half space model of hyperbolic space

\[ \tilde{g} = \frac{(dx^1)^2 + (dx^2)^2 + dy^2}{y^2} \]

Curvature $R[\tilde{g}] = -6$

Henceforth dimension = 3
Asymptotically hyperbolic manifolds

Definition
- Interior of compact manifold with boundary.
- Complete metric $g$
- Boundary geometry locally $\approx$ hyperbolic $\tilde{g}$

Why care?
- Thurston: “most” geometry is hyperbolic
- General relativity
- Complete manifolds are interesting!
Yamabe’s question

Given \((M, g)\) can we rescale the metric to obtain constant scalar curvature?

\[
g \mapsto \phi^4 g \quad \phi > 0
\]

Motivations

▶ 2D has uniformization. Is this the next best thing?
▶ Correct number of degrees of freedom.
▶ Conformal geometry.
In the case of a compact manifold, Yamabe’s problem has a variational formulation.

- Minimize

\[
Q = \frac{\int_M R[\phi^4 g] \, dV_{\phi^4 g}}{\left( \int_M 1 \, dV_{\phi^4 g} \right)^{1/3}} = \frac{\int_M (8|\nabla \phi|^2_g + R[g] \phi^2) \, dV_g}{\left( \int_M \phi^6 \, dV_g \right)^{1/3}} = \frac{E[\phi]}{||\phi||^2_{L^6(M)}}
\]

- Nonlinear eigenvalue problem

\[-8\Delta_g \phi + R[g] \phi = \lambda \phi^5 \quad \text{(Y)}\]

\(\lambda\) is the new scalar curvature
Yamabe problem in AH setting

In the AH setting we obtain

$$- 8\Delta_g \phi + R[g] \phi = -6\phi^5. \quad \text{(Y-AH)}$$

together with the boundary condition $\phi \to 1$.

Main questions:

- existence
- uniqueness
- regularity
Existence and uniqueness

- 1992 Andersson-Chruściel-Friedrich;
- 2018 A-Isenberg-Lee-Stavrov

Existence: Fixed-point argument
Uniqueness: Ratio of two solutions satisfies a linear equation

Use linearization of

$$-8\Delta_g \phi + R[g] \phi + 6\phi^5 = 0 \quad \phi \to 1$$

Take $\phi = 1 + u$ to obtain

$$\Delta_g u - 3u = \underbrace{\frac{1}{8}(6 + R[g])}_f + \underbrace{Q(u) + \frac{1}{8}(6 + R[g])}_N(u)u.$$
Plan is to “integrate in $y$” the linearized form of the equation

$$\Delta_g u - 3u = f + N(u) \quad u \to 0$$

- $f = f_1y + f_2y^2 + f_3y^3 + \ldots$
- If $u = O(y^n)$ then $N(u) = O(y^{n+1})$.
- Integrate one power at a time: $u = u_1y + u_2y^2 + u_3y^3 + \ldots$

Focus on the model problem $\Delta_{\tilde{g}} u - 3u = f + N(u)$
Laplace operator in half-space model I

\[ \tilde{g} = \frac{(dx^1)^2 + (dx^2)^2 + dy^2}{y^2} \]

Dirichlet energy

\[ E = \frac{1}{2} \int |\nabla u|_\tilde{g}^2 dV_{\tilde{g}} = \frac{1}{2} \int y^2 (\text{grad } u) \cdot (\text{grad } u) y^{-3} dx^1 dx^2 dy \]

Using divergence theorem, the first variation is

\[ \dot{E} = \int y^{-1} (\text{grad } u) \cdot (\text{grad } \dot{u}) dx^1 dx^2 dy \]

\[ = - \int y^3 \text{div}(y^{-1} (\text{grad } u)) \dot{u} y^{-3} dx^1 dx^2 dy \]

\[ \Delta_{\tilde{g}} u \]

\[ dV_{\tilde{g}} \]
Linearized equation in half-space model

We computed

$$\Delta_{\bar{g}} u = y^3 \text{div}(y^{-1} \text{grad} \, u)$$

$$= y^3 \partial_y (y^{-2} y \partial_y u) + y^2 (\partial_{x1}^2 u + \partial_{x2}^2 u)$$

$$= y \partial_y (y \partial_y u) - 2y \partial_y u + N(u)$$

Linearized equation is

$$y \partial_y (y \partial_y u) - 2y \partial_y u - 3u = f + N(u)$$

Factored

$$(y \partial_y + 1)(y \partial_y - 3)u = f + N(u)$$
Pop quiz

Compute

\[ y \partial_y (y^n \log y) = ??? \]

Thus

\[ (y \partial_y + 1)(y \partial_y - 3)(????) = \cdots + f_3 y^3 + \ldots \]

Smooth equations might not admit smooth solutions!
What’s going on here?

Change variables $y = e^t$ so that $y \partial_y = \partial_t$.

$$(\partial_t - 3)(\partial_t + 1) u = f + N(u)$$

Homogeneous solutions: $e^{3t} = y^3$ and $e^{-t} = y^{-1}$.

Expand

$$f = f_1 y + f_2 y^2 + f_3 y^3 + \ldots$$
$$\quad = f_1 e^t + f_2 e^{2t} + f_3 e^{3t} + \ldots$$

The $f_3$ term is a resonance!

From the differential equations course... $te^{3t} = y^3 \log y$
For the algebraist

Write as first order system

\[
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f + N(u) \\ 0 \end{pmatrix}
\]

\text{Eigenvalues of } A \text{ are } -1, 3

Duhamel’s formula

\[
U = \int_{s}^{t} \exp(A(t - s))F(s)ds
\]

The algebraic structure of the equations forces log terms upon us.
Results with Isenberg-Lee-Stavrov

▶ Define Banach spaces capturing high interior regularity with limited boundary regularity

▶ Fredholm results for elliptic operators in this category

▶ Applications to Yamabe problem and Einstein constraint equations
Concluding remarks

- Yamabe problem connects parts of our curriculum that don’t often talk to one another – this is fun!

- Hyperbolic geometry is interesting beyond axiomatic geometry

- All those special cases in ODEs are good for something

- Yay!

Thank you!