Chapter 1

Energy and ODEs

In this chapter we show how to use the concept of energy to motivate certain ordinary differential equations.

1.1 Energy

Consider an object with mass m moving about in $\mathbb{R}^3$. We describe the position of the object at time t is given by the function $\mathbf{u}(t) = (x(t), y(t), z(t))$. At any given time t, the velocity vector of the object is

$$\mathbf{v}(t) = \frac{d\mathbf{u}}{dt}(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

(From now on, we frequently drop the explicit dependence on t.)

The ***kinetic energy*** of the object is defined by

$$K = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\left|\frac{d\mathbf{u}}{dt}\right|^2.$$

It is helpful to interpret kinetic energy as the energy of the object associated to motion.

Example 1.1.1

1. The function

$$\mathbf{u} = (1+t, 2+t, 3+t)$$

describes a particle moving along a straight line with constant velocity. The kinetic energy for such a particle is

$$K = \frac{1}{2}m(1^2 + 1^2 + 1^2) = \frac{3}{2}m.$$

2. The function
$$\mathbf{u} = (1 + t^2, 2 + t^2, 3 + t^2)$$
describes a particle moving along the same straight line, but with nonconstant velocity. The kinetic energy for such a particle is
$$K = \frac{1}{2}m((2t)^2 + (2t)^2 + (2t)^2) = 6mt^2.$$

3. A particle moving in a circular trajectory of radius 4 in the xy plane with constant angular velocity is described by
$$\mathbf{u}(t) = (4\cos(t), 4\sin(t), 0).$$
The kinetic energy for such a particle is
$$K = \frac{1}{2}m(4^2) = 8m.$$

In many physical situations, there is also a notion of energy associated to "configuration" or location. This energy is called ***potential energy***, and is given by a function $V = V(x, y, z)$ depending only on the location. The precise formula for potential energy depends on the physical system being considered. It can be helpful to think of $V(\mathbf{u}_2) - V(\mathbf{u}_1)$ as representing the "cost" of moving from location $\mathbf{u}_1$ to location $\mathbf{u}_2$.

Example 1.1.2

1. We say that our object is a ***free particle*** if $V = 0$.

2. The (positive) quadratic potential is
$$V(x, y, z) = \frac{1}{2}k\left(x^2 + y^2 + z^2\right),$$
which we can also write as $V(\mathbf{u}) = \frac{1}{2}k|\mathbf{u}|^2$. Here $k > 0$ is the "strength" of the potential.

3. Suppose there is a larger object of mass M located at the origin.

The gravitational potential energy due to the presence of this object is

$$V(\mathbf{u}) = -\frac{GMm}{|\mathbf{u}|}.$$

(Don't get hung up on the minus sign... it simply ensures that the cost of moving further away is positive.)

The total energy is the sum of the kinetic and potential energy:

$$E = K + V.$$

1.2 Conservation of energy

We now show that the requirement that energy be conserved along the trajectory of our particle gives us a differential equation for the trajectory.

Mathematically, we express the conservation of energy as

$$\frac{d}{dt}E = 0.$$

Direct computation shows that

$$\frac{d}{dt}E = \left(m\frac{d\mathbf{v}}{dt} + \operatorname{grad} V(\mathbf{u})\right) \cdot \mathbf{v},$$

where we have used $\mathbf{v} = \frac{d\mathbf{u}}{dt}$. Thus in order for energy to be conserved, either we need $\mathbf{v} = 0$, which is not especially interesting, or we need

$$m\frac{d\mathbf{v}}{dt} + \operatorname{grad} V(\mathbf{u}) = 0.$$

We write this latter equation as

$$m\frac{d^2\mathbf{u}}{dt^2} = -\operatorname{grad} V(\mathbf{u}). \tag{1.1}$$ NewtonsSecondLaw

The differential equation (1.1) is Newton's second law of motion, where the force is given by $\mathbf{F} = -\operatorname{grad} V(\mathbf{u})$.

Example 1.2.1

1. For a free particle we have $V = 0$. Thus (1.1) becomes

$$m\frac{d^2\mathbf{u}}{dt^2} = 0.$$

From this it is easy to see free particles travel along straight lines with constant velocity.

2. For the quadratic potential we obtain the differential equation

$$m\frac{d^2\mathbf{u}}{dt^2} = -k\mathbf{u},$$

which is the 3D analog of the simple harmonic oscillator.

3. For the gravitational potential we obtain the differential equation

$$m\frac{d^2\mathbf{u}}{dt^2} = -\frac{GMm}{|\mathbf{u}|^3}\mathbf{u}$$

Exercise 1.2.1. *Let* $\mathbf{u} = (x(t), y(t), z(t))$ *so that* $V(\mathbf{u}) = V(x(t), y(t), z(t))$. *Show by direct and detailed computation that*

$$\frac{d}{dt}V(\mathbf{u}) = V(\mathbf{u}) \cdot \frac{d\mathbf{u}}{dt}. \tag{1.2}$$

Exercise 1.2.2. *Suppose that the trajectory is one-dimensional, meaning that* $\mathbf{u}(t) = (x(t), 0, 0)$. *Show that the simple harmonic oscillator equation above reduces to the one we know from differential equations class.*

Exercise 1.2.3. *Show that* $\mathbf{u}(t) = (\cos(\omega t), \sin(\omega t), 0)$ *is a solution to the simple harmonic oscillator equation when* $\omega^2 = k/m$. *Physically interpret this solution.*

1.3 Equilibrium solutions

An equilibrium solution to (1.1) is one where $\mathbf{u}(t)$ is constant. These solutions correspond to points where $\operatorname{grad} V = 0$.

Example 1.3.1

1. For free particles, there exists an equilibrium solution $\mathbf{u}(t) = \mathbf{u}_*$ for every point $\mathbf{u}_*$ in $\mathbb{R}^3$.

2. For the quadradic potential, the only equilibrium solution is $\mathbf{u}(t) = 0$.

3. There are no equilibrium solutions for the gravitational potential.

Exercise 1.3.1. *Consider the one-dimensional problem where*

$$V(x) = (x^2 - 1)^2.$$

Find all of the equilibrium solutions.

1.4 Gradient flows

The gradient flow for a potential function V is the differential equation that describes the trajectory $\mathbf{u}(t)$ that most efficiently moves minimize V. We know that at every location $\mathbf{u}$, the vector $\operatorname{grad} V(\mathbf{u})$ points in the direction in which V increases the most. Thus the vector $-\operatorname{grad} V(\mathbf{u})$ points in the direction in which V decreases the most. Thus the gradient flow for V is given by

$$\frac{d\mathbf{u}}{dt} = -\operatorname{grad} V(\mathbf{u}).$$

Gradient flows tend towards equilibrium points that minimize V, if such equilibrium points exist.

Example 1.4.1

1. For free particles, the gradient flow is

$$\frac{d\mathbf{u}}{dt} = 0.$$

   Since V is constant, there is no motion.

2. For the quadratic potential, the gradient flow is

$$\frac{d\mathbf{u}}{dt} = -k\mathbf{u}.$$

   The solution to this ODE with $\mathbf{u}(0) = \mathbf{u}_0$ is

$$\mathbf{u}(t) = \mathbf{u}_0 e^{-kt}.$$

   Thus all solutions exponentially approach the equilibrium at zero.

3. For the gravitational potential, the gradient flow is

$$\frac{d\mathbf{u}}{dt} = -\frac{GMm}{|\mathbf{u}|^3}\mathbf{u}.$$

This solution approaches zero in finite time; see the exercises.

Exercise 1.4.1. *Suppose that* $\mathbf{u}(t)$ *is a solution to the gradient flow for the gravitational potential; for simplicity set* $GMm = 1$*.*

1. *Show that*
$$\frac{d}{dt}|\mathbf{u}| = -\frac{1}{|\mathbf{u}|^2}.$$
Hint: $|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$*.*

2. *Solve the ODE for* $|\mathbf{u}|$ *in order to conclude that* $|\mathbf{u}| \to 0$ *in finite time.*

Exercise 1.4.2. *Consider the potential function* $V(x) = (x^2 - 1)^2$*.*

1. *Write down the corresponding gradient flow equation.*

2. *Perform a qualitative analysis of the gradient flow equation.*

3. *Which equilibrium solutions will the gradient flow find? Interpret your results in terms of the energy diagram for this potential.*

1.5 Summary

Given an formula $V(\mathbf{u})$ for the potential energy of a system, we obtain

- the associated equations of motion
$$m\frac{d^2\mathbf{u}}{dt^2} = -\operatorname{grad} V(\mathbf{u}),$$
- the equation describing equilibrium solutions
$$\operatorname{grad} V(\mathbf{u}) = 0,$$
and
- the associated gradient flow equation
$$\frac{d\mathbf{u}}{dt} = -\operatorname{grad} V(\mathbf{u}).$$

For the positive quadratic potential $V(\mathbf{u}) = \frac{1}{2}k|\mathbf{u}|^2$ we have the following.

- The equation of motion is the simple harmonic oscillator equation.
- There is a single equilibrium solution $\mathbf{u} = 0$.
- The gradient flow has solutions that exponentially approach the equilibrium solution.