Writing assignment: Gravitation

In this report you explore some simple equations describing the motion of a small planet of mass m as it moves relative to a large star of mass M. We assume the planet moves in the equatorial plane of the planet, so that we can describe the motion of the planet by the functions

$r(t)$, describing the distance from the star to the planet, and

$\theta(t)$, describing the angular motion of the planet.

According to our friends in the Physics Department, the relevant equations are:

$$m\frac{d^2r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 - \frac{GMm}{r^2} \tag{34.3}$$

FirstRadial

$$\frac{d}{dt}\left[mr^2\frac{d\theta}{dt}\right] = 0 \tag{34.4}$$

FirstAngular

Here G is Newton's gravitational constant: $G \approx 6.673 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$ You should take these equations to be physics-given 'facts.'

Your task in this report is to investigate the behavior of the planet predicted by these equations. Before you begin, let me make some remarks:

1. The equation (35.2) implies that the quantity $L = mr^2\frac{d\theta}{dt}$ is conserved; L is called the angular momentum. The fact that it is conserved, means that we can treat L as a parameter (just as we treat the energy as a parameter when studying SHO).

2. Knowing that $L = mr^2\frac{d\theta}{dt}$ is conserved means that we can replace $\frac{d\theta}{dt}$ by $\frac{L}{mr^2}$ in (35.1). The result is an equation which only involves r, though it does have four constant parameters: G, M, m, L. Your plan should be to focus on this "reduced" equation for r because once r is determined, θ can be determined by integration.

3. Although there are four parameters (G, M, m, L), in the reduced equation they only come in two combinations: $a = GM$ and $b = L^2/m^2$. (Note that both a and b are non-negative.) Thus when you are exploring this system, you really only need to think about the two parameters a and b.

4. Finally, since r represents the distance from the star to the planet, we are only interested in the case that $r \geq 0$. (What does it mean for $r \to 0$? What about $r \to \infty$?)

Your report is supposed to be an all-inclusive analysis of the behavior of solutions to the system, based on analyzing the reduced equation for r. As part of your report you should:

1. Explain how to start with the system (35.1)-(35.2) and obtain a reduced equation for r that has two parameters, a and b. Explain how to (physically) interpret the limits $a \to 0$ and $b \to 0$.

2. Show that the reduced equation is actually a Hamiltonian system for some potential function V.

3. Use the potential function to analyze completely the reduced equation, describing possible scenarios, etc.

4. What happens to the potential function in the limit as $a \to 0$ or $b \to 0$? What is the resulting behavior of r?