Lecture 31

Energy diagrams

We now introduce a new tool for visualizing solutions to Hamiltonian systems of the form

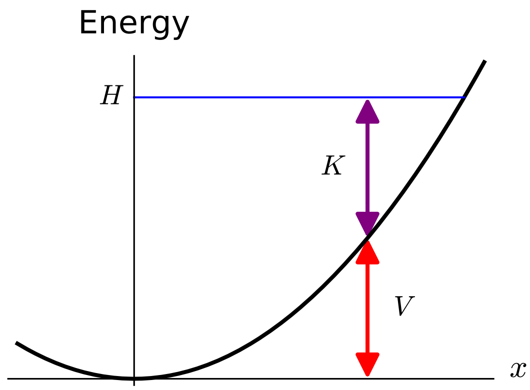
$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -V'(x). \tag{31.1}$$

ED:hamiltonian-system

An ***energy diagram*** is a plot of solutions where the horizontal axis is the spatial position x and the vertical axis is the energy H. Since energy is conserved, the trajectory of a solution to (32.1) in an energy diagram is along a horizontal line, the height of which is the energy of that solution. At each point along the trajectory, we can decompose the height, that is the energh H, in to two components: the kinetic $K = \frac{1}{2}mv^2$ and the potential energy $V = V(x)$. The amount of potential energy at each location is given by the potential function $V(x)$, and thus we can compute the amount of kinetic energy K by simply subtracting.

example:sho-energy-diagram

Example 31.1. *Consider the simple harmonic oscillator, for which the potential function is $V(x) = \frac{1}{2}kx^2$ for some constant k. In the energy diagram below, the solution traverses the blue path at height H. At any given spatial location, the total energy H is the sum of the potential part V and the kinetic part K. The amount of energy that is potential energy is indicated by the potential function $V(x)$, plotted in black. At each location, the vertical "gap" between the potential function and the total amount of energy indicates the amount of kinetic energy K.*



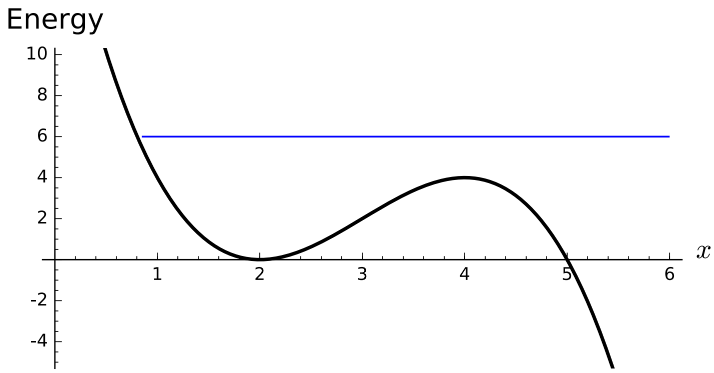
We now make a key observation: the kinetic energy $K = \frac{1}{2}mv^2$ can never be negative. As a consequence, the trajectory of a solution in the energy diagram is always "above" the curve defined by the potential function $V(x)$. Thus a solution will move horizontally across the energy diagram until it reaches the graph of the potential function. At the point where the trajectory reaches this "potential curve" the kinetic energy, and thus the velocity, is zero. Hence the solution instantaneously comes to a halt. The solution, however, does not remain stationary. Consider, for example, the illustration in Example 32.1 above. A solution moving to the right at height H reaches the potential curve at a location where $V'(x)$ is positive. Thus at that moment, we deduce from the differential equation (32.1) that $v' < 0$. Thus the velocity instantaneously decreases from being zero to being negative, at which point the solution begins to move to the left. In this way we see that solutions "reflect" off the graph of the potential function.

Example 31.2. *Consider the potential function* $V(x) = -x(x-2)^2(x-5)$ *from Activity 31.1. The corresponding Hamiltonian system is*

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = 3(x-2)(x-4). \tag{31.2}$$

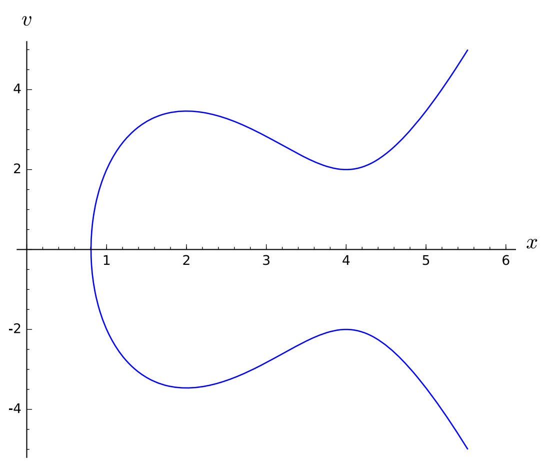
H:cubic-system

The energy diagram for this potential, together with the trajectory of a solution having energy $H = 6$ *is the following:*

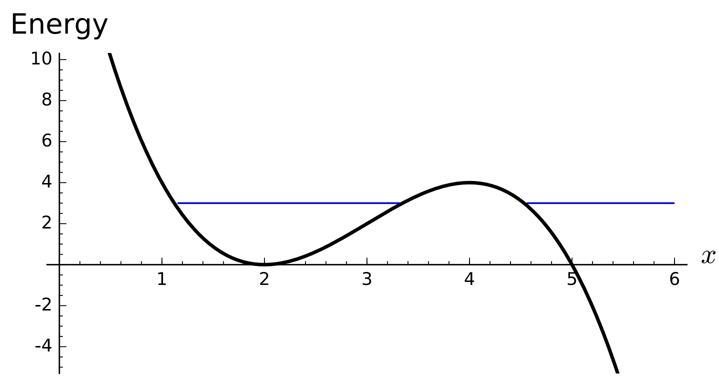


The solution begins at $x = 6$ *and negative* v. *It moves to the left along the line* $H = 6$ *until the trajectory encounters the potential curve at* $x \approx 0.8$. *The solution then reflects off the potential wall and begins to move to the right, still along* $H = 6$, *continuing for all future times.*

We can further understand this solution by plotting the trajectory in the phase plane. Initially, when $x = 6$, *we see that the kinetic energy is large and thus* v *is large in magnitude; since the solution is moving to the left,* v *is negative. As the solution passes over the "bump" at* $x = 4$, *the magnitude of* v *decreases, but then increases again as the solution passes through* $x = 2$. *Then* v *decreases to zero as the solution approaches the potential curve. Following the reflection off the potential curve, the velocity* v *is positive and growing, achieving a local maximum at* $x = 2$ *and then decreases as* x *approaches* 4. *Subsequently,* v *increases without bound. From this we can deduce that the trajectory of this solution in the phase plane is the following:*



Activity 31.1. *Consider again the potential* $V(x) = -x(x-2)^2(x-5)$. *Describe the behavior of two different solutions to* (32.2), *both having with energy* $H = 3$, *that appear in the following energy diagram.*



Draw the corresponding trajectories in the phase plane.

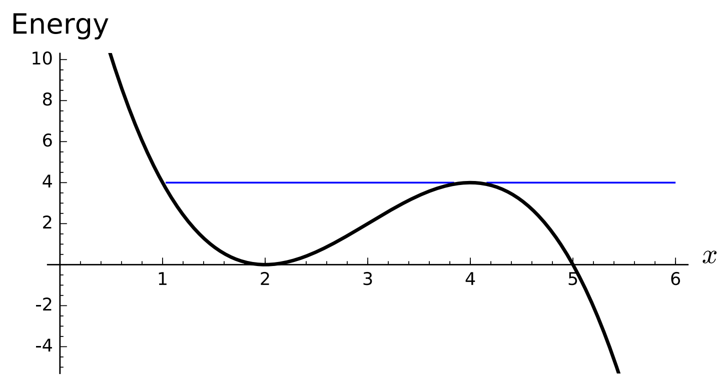
*One of the solutions with energy $H = 3$ is a **bound solution** and one is an **unbounded solution**. Which is which? Use this example to develop a more general definition of these terms.*

In the previous activity and example, we considered solutions to the system (32.2) whose trajectory in the energy diagram either passed well over the "bump" at $x = 4$ or were not high enough to reach close to the top of that bump. However, it is possible that a solution have precisely the right energy to approach the potential curve at a local maximum of the potential function. We now investigate how solutions behave in such a situation.

Suppose that we have a potential function $V(x)$ having a local maximum at $x = x_*$. This means that $V'(x_*) = 0$. Examining the differential equation (32.1), we see that $(x, v) = (x_*, 0)$ is an equilibrium solution. In the energy diagram, this solution simply "sits" at the point $(x, H) = (x_*, V(x_*))$.

We now consider the situation where there is another solution x, v to (32.1) that has energy $H = V(x_*)$ and whose trajectory in the energy diagram is approaching $(x_*, V(x_*))$. Since we cannot have two different solutions to the equation (32.1) meet, we conclude that this second solution x, v approaches the equilibrium point asymptotically, but never reaches it.

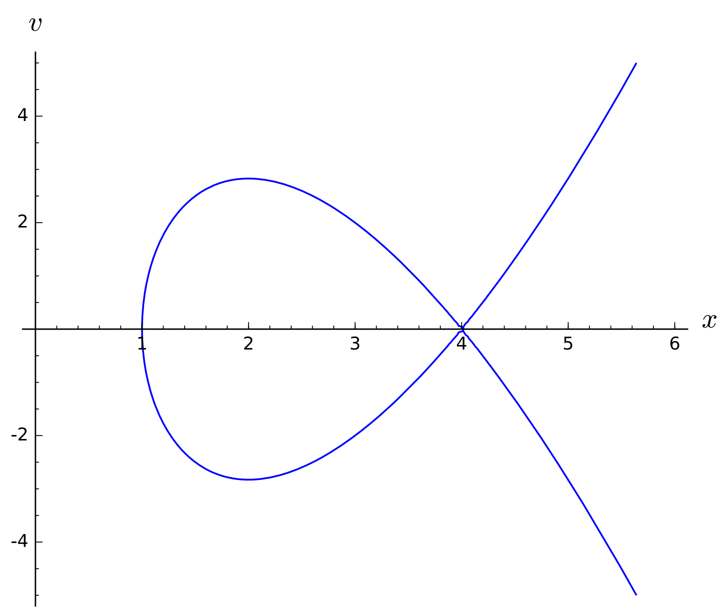
Example 31.3. *Consider once more the potential function $V(x) = -x(x - 2)^2(x - 5)$. Solutions with energy $H = 4$ appear in the energy diagram as follows:*



Those solutions that begin to the left of $x_* = 4$ with negative velocity move to the left until they reflect off the potential curve; then they move to the right, asymptotically approaching $x_* = 4$. Those solutions that initially have $x < 4$ but begin with positive velocity simply asymptote directly to $x_* = 4$.

Solutions to the right of $x_* = 4$ tend towards the equilibrium if their initial velocity is negative, but away from it is their initial velocity is positive.

In the phase diagram, these trajectories are as follows:



We now investigate the equilibrium points for (32.1) more systematically. Our first observation is that *all* of the equilibrium points take the form $(x, v) = (x_*, 0)$, where x_* is a root of $V'(x)$. In other words, the equilibrium points of a Hamiltonian system are precisely the critical points of the potential function.

Activity 31.2. *Suppose* $(x, v) = (0, x_*)$ *is an equilibrium point for* (**??**)*. Show that the linearization of* (32.1) *at* $(0, x_*)$ *is given by*

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -V''(x_*) & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}.$$

Using the results of this activity, we see that the eigenvalues of the linearized system are

$$\lambda = \pm\sqrt{-V''(x_*)}.$$

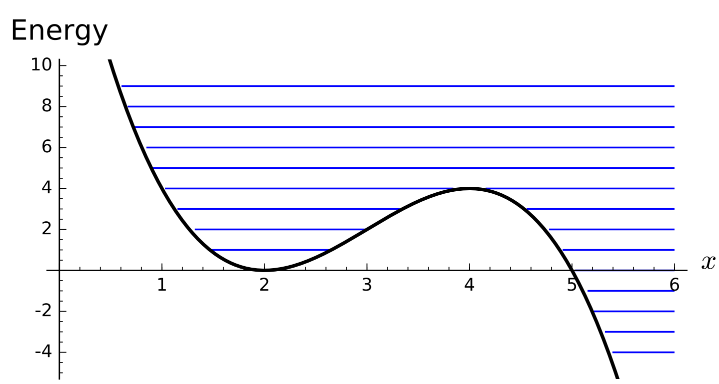
Thus:

- If $V''(x_*) > 0$, then $(x_*, 0)$ is a center-type equilibrium.
- If $V''(x_*) < 0$, then $(x_*, 0)$ is a saddle-type equilibrium.

Of course, if $V''(x_*) = 0$, then we do not learn anything about the stability type of the equilibrium from the linearization. However, it is still the case that if x_* is a local minimum of V then $(x_*, 0)$ is a center and if x_* is a local maximum of V then $(x_*, 0)$ is a saddle. If x_* is a critical point of V but is neither a local minimum nor local maximum, then one obtains a ***degenerate equilibrium***; see Exercise **??**.

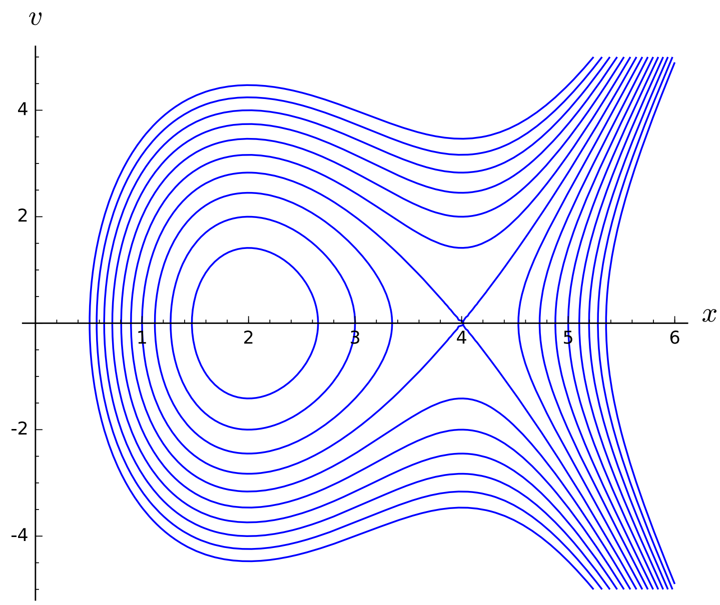
The result of this discussion is that we can entirely understand the solutions to a Hamiltonian system by examining the shape of the potential function. Critical points of the potential function are equilibrium points, and are either saddle-, center-, or degenerate-type. By considering solutions of various energies, we can deduce which initial conditions give rise to bound solutions, and which give rise to unbound solutions. Finally, by tracing each solution through the energy diagram, we can easily deduce the shape of the corresponding solution in the phase plane.

Example 31.4. *Consider one final time the potential function* $V(x) = -x(x-2)^2(x-5)$*. Here we plot the energy diagram with a variety of solution trajectories having different energy levels:*



Notice that we have a center-type equilibrium solution at $x_* = 2$ and a saddle-type equilibrium solution at $x_* = 4$. Solutions with energy $H < 4$ and having $x(0) < 4$ will be bound solutions; these solutions oscillate about the equilibrium $x_* = 2$. All solutions with $H > 4$ or with $H < 4$ but $x(0) > 4$ have $x \to \infty$ as $t \to \pm\infty$. Finally, solutions with $H = 4$ will tend to the equilibrium at $x_* = 4$, unless we have both $x(0) > 4$ and $v(0) > 0$, in which case $x \to \infty$ as $t \to \infty$.

Using this information, we can easily draw the corresponding trajectories in the phase plane:



Exercise 31.1. *Suppose we have the forced oscillator*

$$\frac{d^2x}{dt^2} + 9x = 10$$

1. *Write this as a first order system.*
2. *Show that $H = \frac{1}{2}v^2 + \frac{9}{2}x^2 - 10x$ is a conserved quantity.*
3. *Draw the energy diagram for the equation.*
4. *Draw the phase portrait for the equation.*
5. *Discuss the long-term behavior of solutions to the system, based on your diagram & portrait.*
6. *Find the general solution to the differential equation. Does the behavior match what the pictures predict?*

HW:quartic-potential

Exercise 31.2. *Consider the differential equation*

$$\frac{d^2x}{dt^2} = x^3 - x$$

1. *Write the equation as a first-order system.*
2. *What is the conserved quantity; what is the potential function V?*
3. *Carefully draw the energy diagram. Be sure to include several typical trajectories.*
4. *Carefully draw the phase portrait. Include the paths corresponding to the typical trajectories you put on the energy diagram.*
5. *Analyze the system by making some statements of the sort: Suppose initially _______, then the solution. . .*

Exercise 31.3. *Here you analyze the prototypical case of a degenerate critical point of the potential function.*

- *Work with the potential function $V(x) = x^3$.*
- *Write down the corresponding Hamiltonian system.*
- *Linearize the system at the equilibrium point $(0, 0)$.*
- *Compute the eigenvalues of the matrix for the linearized system, and verify that this situation is indeed "degenerate" in the sense that the linearized system doesn't have two eigensolutions.*

Exercise 31.4. *In this exercise you explore the nullclines of Hamiltonian systems.*

1. *First consider the generic Hamiltonian system (32.1). Describe the nullclines of this system; your description will likely involve the potential function. How can you deduce what the nullclines will be from the plot of the potential function V?*
2. *Find the nullclines of the system in Exercise 32.2.*