Lecture 30

Conservation of energy

Suppose we have a particle moving along the x axis. We describe the motion of the particle with a function $x(t)$ that tells us the location at time t. Physicists have devised two quantities that describe the energy of such a particle:

- The ***kinetic energy K*** is the energy associated to the motion of the particle. The kinetic energy is given by the formula

$$K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2.$$

  If we set $v = \frac{dx}{dt}$ and $p = mv$, then the formula for the kinetic energy can be written as either

$$K = \frac{1}{2}mv^2 \qquad \text{or} \qquad K = \frac{p^2}{2m}.$$

- The ***potential energy V*** is the energy associated to the particle being at a particular location. Consequently, we typically write the potential energy as function $V(x)$, where x is the location of the particle.

The ***total energy H*** is given by

$$H = K + V = \frac{1}{2}mv^2 + V(x). \tag{30.1}$$

Energy:generic

Note that the quantity H is really a function of depends on both $x(t)$, as well its first derivative $x'(t)$. If we want to explicitly indicate this, we can write

$$H[x(t)] = \frac{1}{2}m\left(x'(t)\right)^2 + V(x(t)),$$

or simply

$$H[x] = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x).$$

While the notation $H[x]$ is useful because it reminds us that for each function x we obtain a different quantity $H[x]$, we will typically suppress the dependence x and simply write 'H'.

Since H depends on x, and x depends on t, the energy H can also be viewed as a function of t. We say that the energy H is ***conserved for function x*** if H is a constant function in time. We can mathematically express this ***conservation of energy*** by

$$\frac{d}{dt}H = 0.$$

In classical Newtonian physics, the trajectory of a particle is determined by a differential equation arising from Newton's Second Law,

$$m\frac{d^2x}{dt^2} = F, \tag{30.2}$$

Energy:Newton

where F is an expression describing the various "forces" acting on the particle. While Newton's Second Law is expressed as a second-order equation, it will be more convenient for us to consider its first-order formulation

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = F. \tag{30.3}$$

Energy:Newton2

Example 30.1. *It is useful to keep in mind the example, introduced in §24, of the simple harmonic oscillator equation*

$$m\frac{d^2x}{dt^2} = -kx, \tag{30.4}$$

Energy:SHO

which can be written in first-order form as

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -kx; \tag{30.5}$$

Energy:SHO-system

here k is some positive constant describing the "strength" of the oscillator.

Given a differential equation of the form (31.3), it is natural to ask whether energy is conserved by solutions to that equation. Since the expression for the energy H depends on the potential function $V(x)$, this question can be reformulated as follows:

> *Suppose we have a differential equation of the form (31.3). Does there exist a potential function $V(x)$ such that the corresponding energy $H = \frac{1}{2}mv^2 + V(x)$ is conserved by solutions to the differential equation?*

In order to address this equation, we apply $\frac{d}{dt}$ to H, computing (using the chain rule)

$$\frac{d}{dt}H = mv\frac{dv}{dt} + V'(x)\frac{dx}{dt}.$$

Using that $v = \frac{dx}{dt}$ we see that $\frac{d}{dt}H = 0$ precisely when

$$0 = v\left(m\frac{dv}{dt} + V'(x)\right).$$

Thus one way to ensure that the energy (31.1) is conserved is to require that

$$m\frac{dv}{dt} + V'(x) = 0.$$

In other words, if x and v satisfy the differential equation

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -V'(x) \tag{30.6}$$

Energy:hamiltonian-system

then the energy (31.1) is conserved.

The system (31.6) is called the ***Hamiltonian system*** for potential function $V(x)$. Hamiltonian systems are precisely those systems for which solutions have a conserved energy. Thus if solutions to (31.3) then we see that we must have

$$F = -V'(x)$$

for some potential function $V(x)$.

Example 30.2. *Consider the simple harmonic oscillator equations* (31.5)*. This system takes the form* (31.6) *with potential function*

$$V(x) = \frac{1}{2}kx^2.$$

Thus if x, v are a solution to (31.5)*, then the energy*

$$H = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

is constant.

Activity:cubic-potential

Activity 30.1. *Consider the potential function $V(x) = -(x-2)^2(x-5)$. Find the Hamiltonian system for which the energy*

$$H = \frac{1}{2}v^2 - (x-2)^2(x-5)$$

is conserved.

activity:construct-potential

Activity 30.2. *Consider the second order equation*

$$\frac{d^2x}{dt^2} = -x^2.$$

Show that the corresponding first-order system is a Hamiltonian system. What is a conserved energy for this system?

Notice that when completing Activity 31.2, you actually had some freedom when constructing the potential function. If we are given a differential equation that can be written in the form of a Hamiltonian system, then the potential function is not uniquely determined. We can always add a constant to the potential function and obtain a conserved energy. Mathematically, this is simply a consequence of the fact that the derivative of a constant is zero, which implies that if any particular energy is conserved, then so is that energy plus a constant. In physics, the fact that one can always add a constant to the potential function is in indication that it is the "potential difference" or the relative (rather than absolute) value of the potential that is "physically meaningful."

Thus far we have established that solutions to Hamiltonian systems of the form (31.6) obey conservation energy. We now show how to use this conservation of energy in order to plot the trajectory of solutions in phase space.

First, we note that because the energy H depends only on v and x, we can determine the energy of a solution from the initial conditions

$$x(0) = x_0 \qquad v(0) = v_0.$$

In particular, the energy of a solution with these initial condition is

$$H = \frac{1}{2}mv_0^2 + V(x_0).$$

Once we know the energy H of a given solution, then the conservation of energy implies that at all times we have

$$H = \frac{1}{2}mv^2 + V(x).$$

This implicitly tells us the trajectory that the solution takes through phase space.

Example 30.3. *Suppose x, v is a solution to the simple harmonic oscillator system (31.5) with initial conditions*

$$x(0) = x_0, \qquad v(0) = v_0.$$

The energy of this solution is

$$H = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2. \tag{30.7}$$

Energy:example-sho-energy

At all future times, the solution satisfies

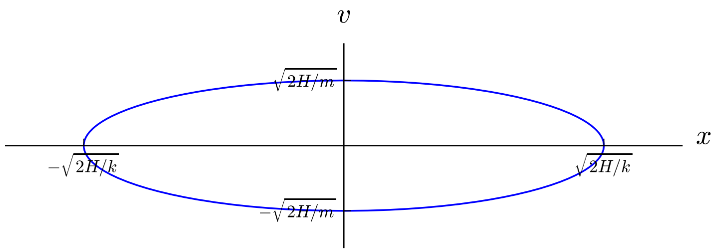
$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2,$$

which we rewrite as

$$v^2 + \frac{k}{m}x^2 = v_0^2 + \frac{k}{m}x_0^2.$$

Thus the solution traverses an ellipse in the phase plane.

With H as in (31.7), the trajectory is as shown in the following diagram:



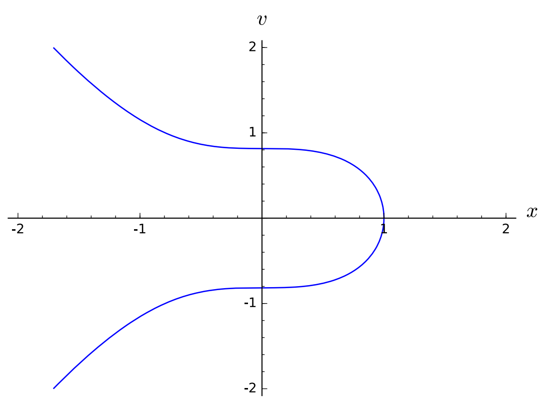
The spatial maximal extent of the solution is $x = \pm\sqrt{2H/k}$, while the maximal velocities achieved by the solution are $v = \pm\sqrt{2H/m}$.

Activity 30.3. *Suppose x is the solution to the equation in Activity 31.2 having initial conditions*

$$x(0) = 1, \qquad x'(0) = 0.$$

What is the trajectory that the solution will follow in phase space?

You should find that the trajectory looks like the following:



We conclude this section with examples of systems that are *not* Hamiltonian systems.

Example 30.4.

1. *Oscillator equations with a non-zero damping term are not Hamiltonian systems. To see this, recall that the first-order system corresponding to the generic oscillator equation is*

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -bv - kx.$$

*Since the right side of the second equation is not a function of x alone it does not take the form (**??**).*

2. *Oscillator equations with time-dependent forcing are not Hamiltonian systems. To see this, we note that the first-order system for the forced simple harmonic oscillator is*

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -kx + f.$$

If f is a function of t, then the right side of the second equation is not a function of x alone, and thus does not take the form (**??**).

→ fix reference

Exercise 30.1. *From* §*24 we know that the general solution to* (31.4) *is*

$$x(t) = \alpha\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + \beta\sin\left(\sqrt{\tfrac{k}{m}}\,t\right).$$

Show by direct computation that the quantity

$$H = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

is constant, thereby confirming that H is conserved.

Exercise 30.2. *Suppose that $x(t)$ is a solution to the initial value problem*

$$4\frac{d^2x}{dt^2} + 9x = 0, \qquad x(0) = 1, \qquad x'(0) = 2.$$

Describe the trajectory of this solution in phase space. What is the maximal velocity attained by the solution? What is the maximal spatial location?

Exercise 30.3. *Consider the differential equation*

$$\frac{d^2x}{dt^2} = x^2.$$

1. *Write the equation in the form of a first-order Hamiltonian system and find the conserved energy.*

2. *Let $x(t)$ be the solution to the initial value problem*

   $$x(0) = 1, \qquad x'(0) = 0.$$

   What is the energy of this solution?

3. *Use conservation of energy to determine the trajectory that the solution takes in phase space.*

Exercise 30.4. *Construct a Hamiltonian system, and an initial condition, for which the corresponding solution traverses the hyperbola*

$$\frac{1}{2}v^2 - \frac{1}{2}x^2 = 50.$$

How many such solutions can you find?

Exercise 30.5. *Explain why the trajectories of solutions to* (31.6) *are always symmetric about the* x *axis.*

Exercise 30.6. *In this problem you study energy for a damped oscillator*

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + x = 0.$$

1. *Write this equation as a first-order system for the variables* x *and* v*.*
2. *Let* $H = \frac{1}{2}v^2 + \frac{1}{2}x^2$*. Supposing that* (x, v) *is a solution to the first order system, show that* $\frac{d}{dt}E = -bv^2$*.*
3. *Conclude that for* $b > 0$ *we have* E *is decreasing. Give a graphical interpretation of this on the phase portrait. How can one use this fact to draw conclusions about the long-term behavior of solutions?*

Exercise 30.7. *In this exercise you explore a system of differential equations that have a conserved quantity that is not an energy.*

1. *Show that* $Q = \frac{x}{y}$ *is a conserved quantity for the system*

$$\frac{dx}{dt} = x^2 - xy \qquad \frac{dy}{dt} = xy - y^2$$

2. *Use the conserved quantity* Q *to draw the phase portrait of the system.*