Lecture 29

General resonance

ch:Resonance-General

→ Also need to revisit repeated eigenvalues here? Or at least connect to the previous stuff on that

As alluded to in the previous section, the mathematical phenomenon of resonance corresponds to a degeneracy in the system. In this section we explore two different types of degeneracies: repeated eigenvalues and degenerate forcing. In both cases, the degeneracy appears when the "usual" methods yield fewer solutions than we were expecting to find. In both cases, we can find the "remaining" solutions in the form of

$$t \cdot (\text{those solutions we were able to find}).$$

I refer to this method as the "t-trick."

The first type of resonance we study is the case of repeated eigenvalues. An example of this was explored in Exercise 28.3. More generally, suppose we have a differential equation of the form

$$\frac{d^2y}{dt^2} - 2\mu\frac{dy}{dt} + \mu^2 y = 0. \tag{29.1}$$

genres:repeated-ODE

The characteristic equation for this ODE is

$$\lambda^2 - 2\mu\lambda + \mu^2 = 0,$$

which we write as

$$(\lambda - \mu)^2 = 0.$$

Thus there is only one eigenvalue, namely $\lambda = \mu$. From this we know that $y_1(t) = e^{\mu t}$ is one solution to (29.1). However, in order to use the superposition principle to find the general solution we need to find a second solution. Motivated by Exercise 28.3, we investigate whether $te^{\mu t}$ is a solution. We compute

$$\frac{d^2}{dt^2}\left[te^{\mu t}\right] - 2\mu\frac{d}{dt}\left[te^{\mu t}\right] + \mu^2\left(te^{\mu t}\right) = 0$$

and thus conclude that $y_2(t) = te^{\mu t}$ is indeed a solution, and therefore that the general solution to (29.1) is

$$y(t) = \alpha e^{\mu t} + \beta te^{\mu t}.$$

Example 29.1. *Suppose we want to solve the initial value problem*

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 3.$$

If we look for solutions of the form $e^{\lambda t}$ we find that λ must satisfy the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0.$$

The only such λ is $\lambda = 4$ and thus we find only the solution e^{4t}. However, the function te^{4t} is also a solution, and thus by superposition the general solution is

$$y(t) = \alpha e^{4t} + \beta te^{4t}.$$

In preparation for imposing the initial conditions, we compute

$$y'(t) = 4\alpha e^{4t} + \beta e^{4t} + 4\beta te^{4t}.$$

Thus the initial conditions reduce to

$$1 = \alpha \qquad \text{and} \qquad 3 = 4\alpha + \beta.$$

Consequently, we find that the solution to the initial value problem is

$$y(t) = e^{4t} - te^{4t}.$$

Activity 29.1. *Find the solution to the initial value problem*

$$9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + y = 7, \qquad y(0) = 1, \qquad y'(0) = 5.$$

The second type of resonance we consider is forced equations for which our "usual guess" for the particular solution does not work. This happens when, for example, the forcing term is of the same term as the homogeneous solution. When this occurs, we make the "educated guess" that the particular solution takes the form $t \cdot y_h(t)$.

Example 29.2. *Consider the differential equation*

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = e^{-2t}.$$

We first address the homogeneous equation

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

by solving the characteristic equation

$$\lambda^2 + 7\lambda + 10 = 0.$$

We find that $\lambda = -2$ and $\lambda = -5$ are solutions, from which we deduce that the homogeneous solution is

$$y_h(t) = \alpha e^{-2t} + \beta e^{-5t}.$$

Notice that the forcing function $f(t) = e^{-2t}$ is of the same form as the homogeneous solution. In particular, if we go looking for a particular solution of the form $y_p(t) = ae^{-2t}$ we obtain the equation

$$0 = e^{-2t},$$

which is a contradiction.

Instead, we use the t-trick, and go looking for a particular solution of the form

$$y_p(t) = ate^{-2t}.$$

Plugging this in to the original equation yields

$$\frac{d^2}{dt^2}\left[ate^{-2t}\right] + 7\frac{d}{dt}\left[ate^{-2t}\right] + 10\left(ate^{-2t}\right) = e^{-2t},$$

which simplifies to

$$3ae^{-2t} = e^{-2t}.$$

Thus we choose $a = 1/3$ and have particular solution

$$y_p(t) = \frac{1}{3}te^{-2t}.$$

Using this, we find the general solution

$$y(t) = \alpha e^{-2t} + \beta e^{-5t} + \frac{1}{3}te^{-2t}.$$

Sometimes we get to use the t-trick twice, as the following activity illustrates!

Activity 29.2. *Find the general solution to the equation*

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^{2t}.$$

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Exercise 29.1. *Find the general solution of the following ODEs:*

1. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 0;$

2. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 1 + t + e^{2t};$

3. $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 9y = 0;$

4. $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = e^{4t}$;

5. $\frac{d^2y}{dt^2} + 9y = \sin(2t)$;

6. $\frac{d^2y}{dt^2} + 9y = 3\cos(3t)$.