Lecture 28

Resonance and oscillators

Resonance1

→ Put trig resonance in to previous section

In this section we continue our study of oscillators with trigonometric forcing. Our goal is to understand what happens when the forcing frequency approaches the natural frequency of the oscillator. To do this, we construct an oscillator equation with natural frequency ω and forcing frequency ω_f as follows:

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega_f t). \tag{28.1}$$

resonance:main

Our plan is the following: We consider (28.1) with initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0 \tag{28.2}$$

resonance:IC

in the situation that $\omega_f \neq \omega$. Then we take the limit as $\omega_f \to \omega$ and see what happens. (The reason for choosing initial conditions (28.2) is that we want to focus attention on the effects of the forcing.)

It is straightforward to see that the homogeneous solution to (28.1) is

$$y_h(t) = \alpha\cos(\omega t) + \beta\sin(\omega t).$$

As in the previous section, we now proceed by looking for a particular solution of the form $y_p(t) = a\cos(\omega_f t)$. Plugging this in to (28.1), obtain

$$(\omega^2 - \omega_f^2)a\cos(\omega_f t) = \cos(\omega_f t). \tag{28.3}$$

resonance:trial-particular

Since we are assuming that $\omega_f \neq \omega$, we obtain the particular solution

$$y_p(t) = \frac{1}{\omega^2 - \omega_f^2}\cos(\omega_f t)$$

and thus see that the general solution is

$$y(t) = \alpha\cos(\omega t) + \beta\sin(\omega t) + \frac{1}{\omega^2 - \omega_f^2}\cos(\omega_f t). \tag{28.4}$$

resonance:general

We now enforce the initial conditions (28.2), computing

$$y'(t) = -\omega\alpha\sin(\omega t) + \omega\beta\cos(\omega t) - \frac{\omega_f}{\omega^2 - \omega_f^2}\sin(\omega_f t).$$

Thus the initial conditions require

$$0 = \alpha + \frac{1}{\omega^2 - \omega_f^2} \qquad \text{and} \qquad 0 = \beta.$$

Consequently, the solution to (28.1) – (28.2) is

$$y(t) = \frac{\cos(\omega_f t) - \cos(\omega t)}{\omega^2 - \omega_f^2}. \tag{28.5}$$

resonance:subcritical-solution

We now want to take the limit of (28.5) as $\omega_f \to \omega$. Notice that both the numerator and the denominator are zero in the limit. Thus it is appropriate to apply l'Hôpital's rule. Keeping in mind that the variable with which we are taking the limit is ω_f, we find that

$$\lim_{\omega_f \to \omega}\left[\frac{\cos(\omega_f t) - \cos(\omega t)}{\omega^2 - \omega_f^2}\right] = \lim_{\omega_f \to \omega}\left[\frac{-t\sin(\omega_f t)}{-2\omega_f}\right] = \frac{t}{2\omega}\sin(\omega t).$$

Thus in the limit as the forcing frequency approaches the natural frequency, the solution approaches a sine wave that oscillates at the natural frequency and has a linearly growing amplitude.

In order to understand what's going on here, it is useful to recall the concept of *beats* from Example 27.3. In that example, we used a trigonometric identity to rewrite the solution in a way that we could interpret as oscillations at the natural frequency with oscillating amplitude. In order to apply that approach here, we use the identity

$$\cos(A) - \cos(B) = 2\sin\left(\frac{B+A}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

in order to write the solution (28.5) as

$$y(t) = \frac{1}{\omega^2 - \omega_f^2}\sin\left(\frac{\omega_f - \omega}{2}t\right)\sin\left(\frac{\omega_f + \omega}{2}t\right).$$

We interpret this last expression as being sinusoidal oscillations of frequency $(\omega_f + \omega)/2$ with periodic amplitude given by

$$\frac{2}{\omega^2 - \omega_f^2}\sin\left(\frac{\omega_f - \omega}{2}t\right).$$

Thus as the forcing frequency approaches the natural frequency, we see that the frequency of the oscillations approaches the natural frequenc; and that the magnitude of the amplitude function increases, while the frequency of the amplitude function decreases. In other words, as ω_f approaches ω, the solution (28.5) consists of increasingly large and slow beats of oscillations at near-natural frequency. In the limit, the first beat "takes over" the solution and we an amplitude function that is simply growing linearly.

The following code demonstrates the limit as $\omega_f \to \omega$ nicely:

```
var('t')
@interact
def f(wf = slider(1.0000001,1.5, default=1.5,
        label='$\omega_f$')):
    solnplot = plot( ( cos(wf*t) - cos(t) )/(1-wf^2), (t,0,120))
    ampplot1 = plot( 2*sin( t*(wf-1)/2)/(1-wf^2)  , (t,0,120),
        linestyle='dashed', color='red',thickness=2)
    ampplot2 = plot( -2*sin( t*(wf-1)/2)/(1-wf^2)  , (t,0,120),
        linestyle='dashed', color='red',thickness=2)
    mainplot = solnplot + ampplot1 + ampplot2
    mainplot.show(ymin=-10,ymax=20)
```

The code can also be accessed via this link.

The picture generated by the code above illustrates the consequences of changing the forcing frequency: As the forcing frequency becomes the closer the natural frequency, the natural response of the system grows dramatically in size, ultimately approaching the linearly growing function

$$y_p(t) = \frac{t}{2\omega}\sin(\omega t). \tag{28.6}$$

resonance:particular-solution

The situation where the forcing frequency is the same as the natural frequency is an example of ***resonance***. Physically, resonance is when the forcing is "tuned" to the natural frequency of the system. This results in oscillations with amplitudes that grow.

Mathematically, we see resonance occur when there is some sort of degeneracy in the system. In the case of the forced oscillator in which the forcing frequency matches the natural frequency, this degeneracy manifests itself in the fact that we do not find a particular solution of the same form as the forcing. Rather, we find a particular solution that is of the form $t \cdot y_h(t)$. This is explored in greater detail in the next section.

Exercise 28.1. *In this problem we consider directly the forced oscillator equation (28.1) when $\omega_f = \omega$:*

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega t). \tag{28.7}$$

resonance:tuned

1. *Show that looking for a particular solution to (28.7) of the form $y_p(t) = a\cos(\omega t)$ yields an equation that cannot be satisfied.*

2. *Show by direct computation that (28.6) is actually a particular solution to (28.7).*

3. *Find the general solution to (28.7).*

HW:cosine-forcing

Exercise 28.2. *Find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 3\cos(2t).$$

HW:first-repeated-root

Exercise 28.3. *In this exercise, we study another type of resonance Consider the equation*

$$\frac{d^2y}{dt^2} - 2b\frac{dy}{dt} + y = 0, \tag{28.8}$$

HW-ExponentialResonance

where b is some parameter with $0 \le b \le 1$.

1. *Find the general solution to (28.8) when $b = 0$.*
2. *Now assume that $0 < b < 1$. Find the solution to (28.8) satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$.*
3. *Show that in the limit as $b \to 1$ we have $y(t) \to te^t$.*
4. *Now set $b = 1$ in the equation (28.8) and verify that te^t is a particular solution.*
5. *Find the general solution to (28.8) in the case when $b = 1$.*
6. *In the case that $b = 1$, we say that (28.8) is resonant. Why is the term "resonant" appropriate in this case?*

Resonance1:RepeatedRoot

Exercise 28.4. *Find the general solution to*

$$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = 0.$$