Lecture 27

Oscillators with trigonometric forcing

In this section we address oscillators with trigonometric forcing. If there is no damping present in the oscillator, then we can proceed essentially as in the previous section.

Example 27.1. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(3t).$$

We easily see that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

Since the forcing function is a cosine function with frequency 3*, we look for a particular solution of the form* $y_p(t) = a\cos(3t)$*. Plugging this in to the equation yields*

$$-9a\cos(3t) + 4a\cos(3t) = \cos(3t).$$

From this we deduce that $a = -1/5$ *and thus we have*

$$y_p(t) = -\frac{1}{5}\cos(3t).$$

The general solution is therefore

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{5}\cos(3t).$$

In order to understand this solution, it is helpful to make use of the trigonometric identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

to write

$$\cos(3t) = \cos(2t)\cos(t) - \sin(2t)\sin(t).$$

Thus the general solution can be written

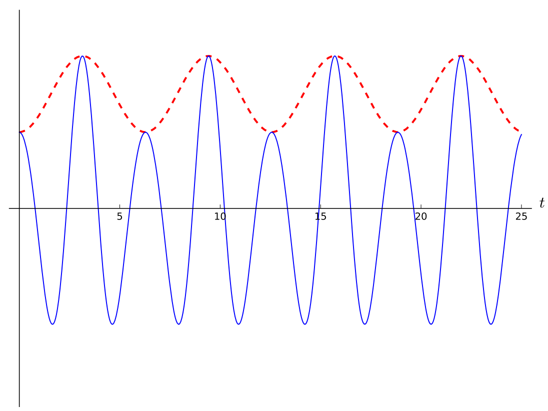
$$y(t) = \left(\alpha - \frac{1}{5}\cos(t)\right)\cos(2t) + \left(\beta + \frac{1}{5}\sin(t)\right)\sin(2t).$$

We interpret the function

$$\left(\alpha - \frac{1}{5}\cos(t)\right)\cos(2t)$$

as the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{5}\cos(t)$*. Thus we see that the solutions* $y(t)$ *oscillate at the frequency associated to the homogeneous equation, but that the amplitudes of these oscillations themselves oscillate at the frequency that is the difference between the forcing frequency and the homogeneous frequency.*

A typical solution looks something like the following:



Here the red dashed line is the graph of the function $\alpha - \frac{1}{5}\cos(t)$*, and the blue solid line is the graph of the function* $(\alpha - \frac{1}{5}\cos(t))\cos(2t)$*.*

The previous example illustrates some interesting interplay between the frequency at which the homogeneous solution oscillates and the frequency of the forcing. In order to discuss this interplay, we introduce some terminology. Let ω_h be the frequency at which solutions to the homogeneous equation oscillate; we call ω_h the ***natural*** frequency. Furthermore, let ω_f be the frequency present in the forcing function; we call ω_f the ***forcing frequency***. The previous example shows that we can expect solutions to a forced oscillator with trigonometric forcing to oscillate at frequency ω_h, and that we can expect the amplitude of these oscillations to themselves oscillate with frequency $\omega_f - \omega_h$.

The following two examples illustrate what happens when the forcing frequency is either very far from, or very close to, the natural frequency.

Example 27.2. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(30t).$$

In this case, the forcing frequency $\omega_f = 30$ *is very far from the natural frequency* $\omega_h = 2$.

As in the previous example, the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We look for a particular solution of the form $y_p(t) = a\cos(30t)$. *Plugging this in to the equation we obtain*

$$-90a\cos(30t) + 4a\cos(30t) = \cos(30t).$$

Thus we choose $a = -1/86$ *and obtain the particular solution*

$$y_p(t) = -\frac{1}{86}\cos(30t).$$

Consequently, the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{86}\cos(30t).$$

Using the trigonometric identity

$$\cos(30t) = \cos(28t)\cos(2t) - \sin(28t)\sin(2t)$$

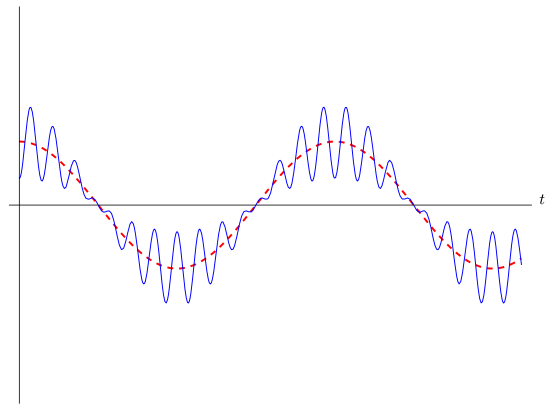
we express the general solution as

$$y(t) = \left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t) + \left(\beta + \frac{1}{86}\sin(28t)\right)\sin(2t)$$

We can interpret the function

$$\left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t)$$

to be the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{86}\cos(28t)$. *Notice that the frequency at which the amplitude is changing is much faster than the natural frequency of the oscillator. Thus a typical solution looks something like the following:*



Here the graph of the function

$$\left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t)$$

is shown with a solid blue curve, while the graph of the function $\alpha\cos(2t)$ *is shown with a red dashed curve*

example:beats

Example 27.3. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(2.1t).$$

In this case, the forcing frequency $\omega_f = 2.1$ *is very close to the natural frequency* $\omega_h = 2$.

As in the previous example, the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We look for a particular solution of the form $y_p(t) = a\cos(2.1t)$. *Plugging this in to the equation we obtain*

$$-4.41a\cos(2.1t) + 4a\cos(2.1t) = \cos(2.1t).$$

Thus we choose $a = -1/86$ *and obtain the particular solution*

$$y_p(t) = -\frac{1}{0.41}\cos(2.1t).$$

Consequently, the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{0.41}\cos(2.1t).$$

Using the trigonometric identity

$$\cos(2.1t) = \cos(0.1t)\cos(2t) - \sin(0.1t)\sin(2t)$$

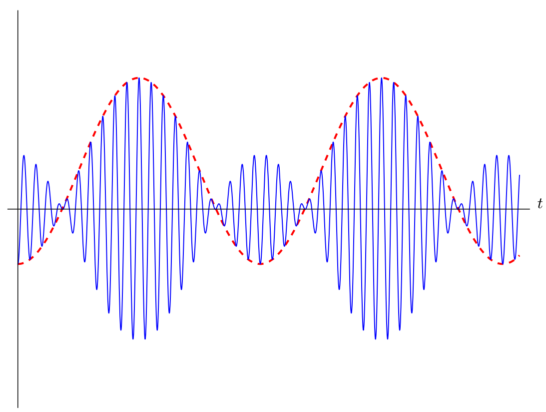
we express the general solution as

$$y(t) = \left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t) + \left(\beta + \frac{1}{0.41}\sin(0.1t)\right)\sin(2t)$$

We can understand the function

$$\left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t)$$

as the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{0.41}\cos(0.1t)$. *Notice that the frequency changes of the amplitude is much smaller than the frequency of the natural oscillations. This gives rise to solutions that look like the following:*



Here the graph of the function

$$\left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t)$$

is shown as a solid blue curve, while the graph of the amplitude function $\alpha - \frac{1}{0.41}\cos(0.1t)$ *is shown as a dashed red curve. The clusters of oscillations shown in this graph are known as* ***beats****, and are typical of cases when the forcing frequency is very close to the natural frequency.*

Beats are frequently used by musicians to tune stringed instruments: Two strings being "in tune" mean that they oscillate at the same natural frequency. If the two strings are very slightly out of tune, then playing one of the strings will have the effect of forcing the other string at a frequency nearby to its natural frequency, resulting in the formation of beats in the vibrations of the second string.

In all of the previous examples, we were able to find a particular solution that was a multiple of the forcing function. Unfortunately, as the following example illustrates, that does not work when there is damping present. However, we are still able to guess a particular solution by considering both cosine and sine functions.

Example 27.4. *Consider the equation*

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 4y = \cos(3t),$$

which describes an underdamped oscillator with periodic forcing.

The characteristic equation for the homogeneous equation is

$$\lambda^2 + \lambda + 4 = 0,$$

which has solutions

$$\lambda_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}\, i.$$

Thus the homogeneous solution is

$$y_h(t) = \alpha e^{-t/2} \cos\left(\frac{\sqrt{15}}{2}\, t\right) + \beta e^{-t/2} \sin\left(\frac{\sqrt{15}}{2}\, t\right).$$

In order to find a particular solution, we guess that $y_p(t) = a\cos(3t) + b\sin(3t)$*. Plugging this in to the original equation gives us*

$$\{-9a + 3b + 4a\}\cos(3t) + \{-9b - 3a + 4b\}\sin(3t) = \cos(3t).$$

Thus in order to obtain a solution we need

$$-5a + 3b = 1 \qquad \textit{and} \qquad -3a - 5b = 0.$$

Thus we need $a = -5/34$ *and* $b = 3/34$*. The resulting particular solution is*

$$y_p(t) = -\frac{5}{34}\cos(3t) + \frac{3}{34}\sin(3t).$$

Notice that when guessing our particular solution, we cannot have $b = 0$*. This means that we could not have constructed a particular solution with only* $\cos(3t)$ *in it. (It is a good exercise to try... what goes wrong?)*

Force-MidBegin

Exercise 27.1. *The equation*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t)$$

models frictionless oscillations with periodic forcing.

1. *Find the general solution to the homogeneous equation*

$$\frac{d^2y}{dt^2} + 9y = 0.$$

2. *Solve the (homogeneous) initial value problem*

$$\frac{d^2y}{dt^2} + 9y = 0, \qquad y(0) = 0, \quad y'(0) = 5.$$

3. *Find a particular solution of to the inhomogeneous equation*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t).$$

4. *Find the general solution of the equation*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t).$$

5. *Solve the IVP*

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t), \qquad y(0) = 0, \quad y'(0) = 5.$$

6. *Graph the solution of the IVP in the ty-plane, paying particular attention to long-term behavior of the graph.*

7. *Describe (in words!) the effect the forcing has on the oscillator.*

Exercise 27.2. *Repeat Exercise 27.1 for the initial value problem*

$$\frac{d^2y}{dt^2} + 16y = 7\sin(3t) \qquad y(0) = 0, \quad y'(0) = 0.$$

Force-MidBegin-Repeat2

Exercise 27.3. *Repeat Problem 27.1 for the initial value problem*

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin(t) \qquad y(0) = 0, \quad y'(0) = 0.$$

Exercise 27.4. *Solve the initial value problem*

$$\frac{d^2y}{dt^2} + 16y = \cos 25t, \qquad y(0) = 0, \qquad y'(0) = \frac{1}{100}.$$

Plot the solution and describe its characteristic features.