Lecture 26

Oscillators with forcing

In the previous sections we developed a good understanding of solutions to homogeneous equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{26.1}$$

inhom:generic-hom

Our goal now is to understand what happens when we introduce forcing. Thus we study equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f, \tag{26.2}$$

inhom:generic

where f is some function of t, and where a, b, c are constants.

In order to handle forcing, we introduce the following "generalized superposition principle."

Generalized superposition principle Suppose $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous equation (26.1) and that $y_p(t)$ is a solution to the inhomogeneous equation (26.2). Then

$$y(t) = \alpha y_1(t) + \beta y_2(t) + y_p(t)$$

is a solution to (26.2) for any constants α and β.

The generalized superposition principle can be physically interpreted as follows: Suppose we have an inhomogeneous equation of the form (26.2). Then the general solution

$$y_h(t) = \alpha y_1(t) + \beta y_2(t) \tag{26.3}$$

to the associated homogeneous equation (26.1) is the "natural response" of the system, in the absence of any external forcing. The function $y_h(t)$ is often called the ***homogeneous solution***. The function $y_p(t)$ is an additional contribution to $y(t)$ that represents the "response" of the system to the external forcing. The function $y_p(t)$ is usually called a ***particular solution***.

The generalized superposition principle suggests an approach for finding the general solution to equations of the form (26.2):

1. Find the general solution $y_h(t)$ to the homogeneous equation,
2. Find a particular solution $y_p(t)$ to the inhomogeneous equation,
3. Construct the general solution $y(t) = y_p(t) + y_h(t)$ to the inhomogeneous equation.

Since homogeneous equations are well understood, the general superposition principle means that if we can find one solution to an inhomogeneous equation, then we can easily find them all!

However, we still have to find one solution to (26.2). Unfortunately, the most efficient method that mathematicians have been able to come up with thus far is. . . educated guess and check. The guiding principle is this:

> *Look for a particular solution $y_p(t)$ that is the same "type" of function that $f(t)$ is.*

If you find this unsatisfying, you have good company. There is general theory out there about constructing particular solutions. But the educated guess-and-check method is so much more efficient for the simple cases we'll cover in this class that it simply isn't worth it to get out the fancy theory at this point.

The following list of examples demonstrates how the educated guess and check method works.

Example 26.1. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 7.$$

First we consider the associated homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation is $\lambda^2 + 4 = 0$, and thus the eigenvalues are $\lambda = \pm 2i$. This implies that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We now try to guess that a particular solution. Since the forcing function f is constant, we guess that the particular solution takes the form $y_p(t) = a$ for some constant a. Plugging this in to the original equation yields

$$0 + 4a = 7.$$

Thus we obtain a solution $y_p(t) = 7/4$.

Assembling the pieces, we see that the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) + \frac{7}{4}.$$

Example 26.2. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 7t.$$

First we consider the associated homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation is $\lambda^2 + 4 = 0$, and thus the eigenvalues are $\lambda = \pm 2i$. This implies that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We now try to guess that a particular solution. Since the forcing function f is a linear function, we guess that the particular solution takes the form $y_p(t) = a + bt$ for some constants a, b. Plugging this in to the original equation yields

$$0 + 4(a + bt) = 7t.$$

We rearrange this to

$$4a + (4b - 7)t = 0.$$

In this last equation, we have equality in the sense of functions. Thus we must have $a = 0$ and $b = 7/4$. Thus the particular solution is

$$y_p(t) = \frac{7}{4}t$$

and the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) + \frac{7}{4}t.$$

Example 26.3. *Let's analyze the solutions to the equation*

$$\frac{d^2y}{dt^2} + 4y = e^{2t}.$$

First we address the homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

From the previous examples, we know that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We now look for a particular solution. Since the forcing function is exponential with growth rate 2, we look for a particular solution of the form

$$y_p(t) = ae^{2t}$$

where a is some constant. Plugging this in to the differential equation yields

$$4ae^{2t} + 4ae^{2t} = e^{2t}.$$

It is easy to see that choosing $a = 1/8$ leads to a particular solution

$$y_p(t) = \frac{1}{8}e^{2t}$$

and thus the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) + \frac{1}{8}e^{2t}.$$

Notice that as $t \to \infty$, we have $y_p(t) \gg y_h(t)$ and thus the forced response dominates behavior far in the future. As $t \to -\infty$ we have $y_p(t) \ll y_h(t)$ and thus the natural response dominates far in the past.

Activity 26.1. *Find the general solution to the differential equation*

$$\frac{d^2y}{dt^2} + 4y = e^{-2t}.$$

In what regime is y_h dominant? In what regime is y_p dominant?

Finally, we give an example in which we solve an initial value problem.

Example 26.4. *Let's solve the initial value problem*

$$\frac{d^2y}{dt^2} + 16y = 2t + 1, \qquad y(0) = 2, \quad y'(0) = 0.$$

To accomplish this, we first find the general solution.

The homogeneous equation is

$$\frac{d^2y}{dt^2} + 16y = 0,$$

which has characteristic equation $\lambda^2 + 16 = 0$. Thus the eigenvalues are $\lambda = \pm 4i$ and the homogeneous solution is

$$y_h(t) = \alpha\cos(4t) + \beta\sin(4t).$$

Since the forcing function is linear, we look for a particular solution of the form $y_p(t) = a + bt$. Plugging this in to the original equation yields

$$16(a + bt) = 2t + 1.$$

This is satisfied if we choose $a = 1/16$ and $b = 1/8$. Thus the particular solution is

$$y_p(t) = \frac{1}{16} + \frac{1}{8}t$$

and the general solution is

$$y(t) = \alpha \cos(4t) + \beta \sin(4t) + \frac{1}{16} + \frac{1}{8}t.$$

We now impose the initial conditions. The condition that $y(0) = 2$ *becomes*

$$2 = \alpha + \frac{1}{16}.$$

Computing

$$y'(t) = -4\alpha \sin(4t) + 4\beta \cos(4t) + \frac{1}{8}$$

we see that the condition $y'(0) = 0$ *becomes*

$$0 = 4\beta + \frac{1}{8}.$$

Thus $\alpha = 31/16$ *and* $\beta = -1/32$*, which implies that the solution to the initial value problem is*

$$y(t) = \frac{31}{16}\cos(4t) - \frac{1}{32}\sin(4t) + \frac{1}{16} + \frac{1}{8}t.$$

Exercise 26.1. *Consider the non-homogeneous equation:*

$$\frac{d^2y}{dt^2} + 9y = 9.$$

This equation arises from studying a frictionless oscillator with constant forcing.

1. *Find a particular solution of this equation.*

2. *Find the homogeneous solution.*

3. *Based on the above find the general solution of the equation.*

4. *Solve the IVP*

$$\frac{d^2y}{dt^2} + 9y = 9, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 3.$$

5. *Graph the solution of the IVP in the ty-plane, paying particular attention to long-term behavior of the graph.*

6. *How would you in words describe the effect the forcing has on the oscillator?*

Exercise 26.2. *Repeat Problem 26.1 for the differential equation* $\frac{d^2y}{dt^2} + 9y = 10e^{-t}$ *and IVP*

$$\begin{cases} \frac{d^2y}{dt^2} + 9y = 10e^{-t}, \\ y(0) = 0, \quad \frac{dy}{dt}(0) = -7. \end{cases}$$

Exercise 26.3. *Repeat Problem 26.1 for the equation* $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1$ *and the IVP*

$$\begin{cases} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1, \\ y(0) = 0, \quad \frac{dy}{dt}(0) = 0. \end{cases}$$

Note: this equation represents an oscillator with friction.

Exercise 26.4. *Repeat Problem 26.1 for the equation* $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1 + 3e^t$ *and the IVP*

$$\begin{cases} \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1 + 3e^t, \\ y(0) = 2, \quad \frac{dy}{dt}(0) = 1. \end{cases}$$

Exercise 26.5. *Solve the initial value problem*

$$\begin{cases} \dfrac{d^2y}{dt^2} + 16y = e^{-\frac{t}{10}} \\ y(0) = \dfrac{110}{1601}, \quad y'(0) = \dfrac{-10}{1601} \end{cases}$$

Plot the solution and describe its characteristic features.

Exercise 26.6. *Find general solutions of the following differential equations.*

1. $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 6y = 4e^{-2t}$;

2. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + y = 1 + e^{-t}$.

Exercise 26.7. *Find the general solution of the equation*

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 2t + 1.$$

Exercise 26.8. *Devise a recipe for finding a forced response of the oscillator with polynomial forcing* $f(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t + a_0$.