Lecture 24

Modeling oscillations

section:modeling-oscillations

In this chapter we study equations that model one-dimensional oscillations. Let's begin by with some formulas borrowed from physics. At this stage, we won't concern ourselves with how the physicists obtained these formulas, though that is a very interesting subject. Rather, we accept that the physicists have good reasons for believing that the formulas are relevant and proceed to mathematically analyze the consequences of accepting the formulas.

Our first formula comes from Newton, who asserted that objects are subject to "forces" and that the change in momentum of an object is equal to the total force acting upon the object. To convert this in to a formula, we first suppose that we have an object of mass m that is only able to move in one direction, which we call the y-direction. We describe the movement of the object by the function $y(t)$, which tells us the location of the "center of mass" of the object at each time t. The ***momentum*** of the object is defined to be

$$p = m\frac{dy}{dt},$$

and is thus a function as well. (Technically, momentum is a vector. However, since we are only considering motion in one direction the momentum vector has only one component. Thus we are able to describe the momentum by a single function.) We can express Newton's assertion that the change in momentum is equal to the force by the formula

$$F = \frac{dp}{dt}, \tag{24.1}$$

Newton

which is known as "Newton's Second Law." It is common to rewrite (24.1) by computing

$$\frac{dp}{dt} = m\frac{d^2y}{dt^2}$$

and labeling the acceleration $\frac{d^2y}{dt^2}$ with the letter a; thus (24.1) becomes the more famous formula

$$F = ma.$$

However, since this is a differential equations course it is more convenient for us to keep the derivatives explicitly present. Thus we either write

$$F = \frac{dp}{dt} \qquad \text{and} \qquad p = m\frac{dy}{dt}$$

or we write

$$F = m\frac{d^2y}{dt^2}.$$

The second formula from physics that we need is due to Hooke (who, incidentally, corresponded with Newton engaged in a priority dispute with him concerning the inverse square law of gravitation). Suppose our object of mass m is attached to a spring in such a way that the object will remain at rest if placed at $y = 0$, but that the object will feel a force from the spring if displaced from location $y = 0$. Hooke's Law is the postulation that the force of the spring upon the mass is proportional to the displacement. Mathematically, we express this assertion as

$$F_{\text{spring}} = -ky,$$

where $k > 0$ is known as the "spring constant."

Together, Newton's Second Law and Hooke's Law combine to give us the differential equation

$$m\frac{d^2y}{dt^2} = -ky,$$

which is known as the ***simple harmonic oscillator***.

The simple harmonic oscillator is a basic model of oscillations. More complicated models can be constructed by including terms that account for external forces and for friction. The class of oscillator models we consider in this course take the form

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f, \tag{24.2}$$

generic-oscillator-prelim

where the $-b\frac{dy}{dt}$ is a simple model for ***frictional damping*** and f describes any external ***forcing*** present. Here b is a constant, but f might be a function of t.

Example 24.1. *Near the surface of the earth, the force of gravity upon an object of mass m is given by $f = -mg$, where $g \approx 9.8\, m/s^2$ and the negative sign is due to the fact that the force is downward. Thus the height y of an oscillating mass hanging from a spring can be modeled with the differential equation*

$$m\frac{d^2y}{dt^2} = -ky - mg.$$

There are two "standard" ways to write the generic oscillator equation (24.2). The first standard way to write it is to put all the y terms on the left, leaving the forcing on the right:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f. \tag{24.3}$$

generic-oscillator

Equations of the form (24.3) are called ***constant-coefficient, second-order*** because they involve second derivatives of the unknown y and because the coefficients m, b, k are constants.

The second standard way to write (24.2) is as a first-order system. We can do this by introducing a new variable, the velocity

$$v = \frac{dy}{dt}. \tag{24.4}$$

define-v

If we replace every instance of $\frac{dy}{dt}$ with v, then (24.2) becomes

$$m\frac{dv}{dt} = -ky - bv + f. \tag{24.5}$$

generic-replacement

Thus (24.2) can be written as the system

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = -\frac{k}{m}y - \frac{b}{m}v + \frac{f}{m}. \tag{24.6}$$

generic-oscillator-first-order

As an aside, we remark that in classical mechanics it is common to work in terms of the momentum p rather than the velocity v. In this case, the system of equations becomes

$$\frac{dy}{dt} = \frac{1}{m}p, \qquad \frac{dp}{dt} = -ky - \frac{b}{m}p + f.$$

Of course, the two are mathematically equivalent... and in fact we could easily re-do the rest of this section using the p variable. But we'll stick with the variable v in these notes.

We will frequently go back and forth between the second-order formulation (24.3) and the first order formulation (24.6). We illustrate this with the following example.

Example 24.2. *The simple harmonic oscillator can be written as either*

$$m\frac{d^2y}{dt^2} + ky = 0$$

or as

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = -\frac{k}{m}y.$$

The first order formulation we can write as

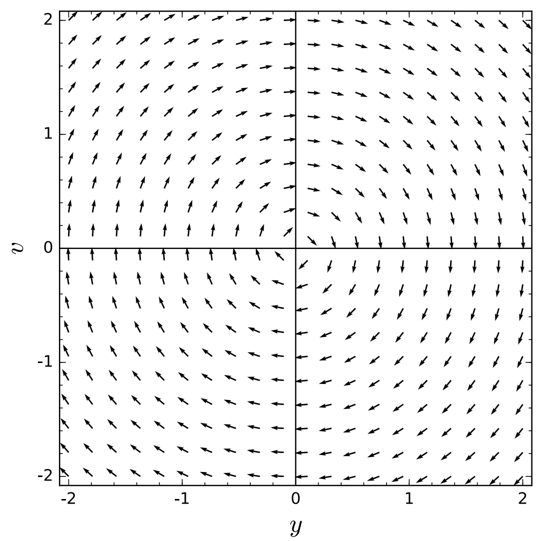
$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}.$$

We can easily solve the first-order system using eigenstuff. First, we compute the eigenvalues to be

$$\lambda_{\pm} = \pm i\sqrt{\frac{k}{m}}.$$

Thus there is a center equilibrium at $(y, v) = (0, 0)$. It is easy to verify that the direction of rotation is clockwise; consequently the phase plane picture looks something like this:

→ **update graphic?**



We can easily read off the phase plane that the function $y(t)$ is periodic. Furthermore, when y is large, its derivative $v = \frac{dy}{dt}$ is small and vice-versa. Thus we learn something about solution to the second-order equation by looking at the phase portrait for the first-order system!

We now use the first-order system to construct solutions to the second-order equation. Focusing on λ_+ we choose eigenvector

$$\begin{pmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{pmatrix}.$$

Thus we obtain the complex eigensolution

$$Y_+(t) = e^{i\sqrt{\frac{k}{m}}\,t} \begin{pmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ -\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix} + i \begin{pmatrix} \sin\left(\sqrt{\frac{k}{m}}\,t\right) \\ \sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix}.$$

From this we obtain two independent real solutions

$$\begin{pmatrix} \cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ -\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} \sin\left(\sqrt{\frac{k}{m}}\,t\right) \\ \sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix}.$$

Thus the general solution to the first order formulation of the simple harmonic oscillator is

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \alpha \begin{pmatrix} \cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ -\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix} + \beta \begin{pmatrix} \sin\left(\sqrt{\frac{k}{m}}\,t\right) \\ \sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix}.$$

We can translate these results back to second-order equation. Reading off the first components of the vector we see that the general solution for y is given by

$$y(t) = \alpha \cos\left(\sqrt{\frac{k}{m}}\, t\right) + \beta \sin\left(\sqrt{\frac{k}{m}}\, t\right).$$

Actually, we can learn one more thing about the second order equation for the simple harmonic oscillator by looking at the first-order system. We know that an appropriate initial condition for the first-order system takes the form

$$\begin{pmatrix} y(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

for some constants y_0 and v_0. In fact, we can easily see that the solution to the corresponding initial value problem is obtained by choosing $\alpha = y_0$ and $\beta = v_0\sqrt{\frac{m}{k}}$. From this we learn that the appropriate initial conditions for the second-order equation take the form

$$y(0) = y_0 \qquad \text{and} \qquad \frac{dy}{dt}(0) = v_0,$$

and that the solution to the corresponding initial value problem is given by

$$y(t) = y_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + v_0\sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}\, t\right).$$

In general, given a second order equation of the form (24.3) we can form an associated first-order system of the form (24.6). The previous example illustrates how to use the phase diagram of the first order system in order to describe solutions to the second-order equation. We also see that the appropriate initial conditions for a second order equation involve specifying both the initial value and initial value of the first derivative of the unknown. Physically this means specifying the initial position and the initial velocity.

Example 24.3. *Consider the second-order equation*

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0.$$

This equation is equivalent to the first-order system

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}.$$

The characteristic equation for the matrix in this system is

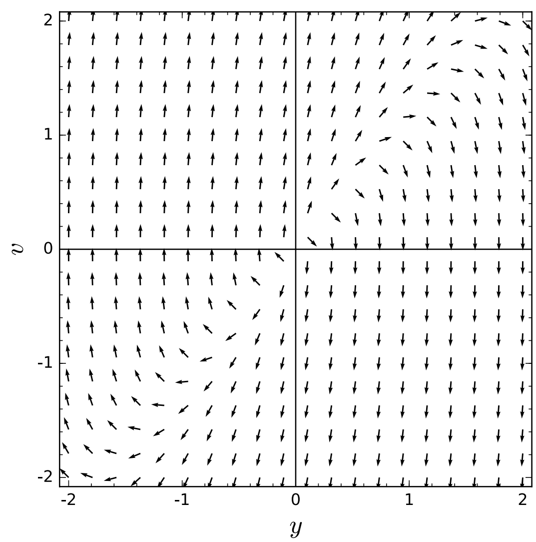
$$\lambda^2 - 5\lambda + 6 = 0$$

and thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

We compute the corresponding eigenvectors to be

$$\begin{pmatrix}1\\2\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}1\\3\end{pmatrix}.$$

Thus the phase portrait for the system looks like → **update graphic?**



From this we see that if $v_0 > 2y_0$ then we have $v, y \to \infty$ as $t \to \infty$, while if $v_0 < y_0$ then $v, y \to -\infty$. If $v_0 = 2y_0$, then the $v, y \to \pm\infty$ with the sign equal to the sign of y_0.

Since the general solution to the first order system is

$$\begin{pmatrix}y(t)\\v(t)\end{pmatrix} = e^{2t}\begin{pmatrix}1\\2\end{pmatrix} + e^{3t}\begin{pmatrix}1\\3\end{pmatrix}$$

we may deduce that the general solution to the original second order equation is

$$y(t) = \alpha e^{2t} + \beta e^{3t}.$$

This last example suggests a close connection the general solution to the second order equation and the eigenvalues of the matrix associated to the associated first order system. This connection is discussed in more detail in the next section.

Activity 24.1. *Write the second order equation*

$$3\frac{d^2y}{dt^2} + 13\frac{dy}{dt} - 10y = 0$$

as a first order system. Use the first order system to determine how solutions to the original second order equation behave.

Hom-First

Exercise 24.1. *Re-write the following second order equations as the first order system. Then use* `Sage` *to generate the phase portraits of the oscillators. Use the phase portrait to discuss the behavior of a 'typical' solution* $y(t)$*.*

1. $\dfrac{d^2y}{dt^2} + 4y = 0$;
2. $\dfrac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 3$;
3. $\dfrac{d^2y}{dt^2} + 3y^2 = 3$.

Home-Second

Exercise 24.2. *In this problem you study the equation*

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 4y = 0,$$

for various values of $b \geq 0$*.*

1. *Write this equation as a first-order system.*
2. *Use the following* `Sage` *code to explore the phase portrait for various values of* b*:*

```
var('x,y')
@ interact
def _(b = (0,(0,5)) ):
    Field = [y,-4*x-b*y]
    Fieldplot = plot_vector_field(vector(Field)/vector(Field).
        norm(),(x,-2,2),(y,-2,2))
    Fieldplot.show(figsize=[5,5],axes_labels=['$y$','$v$'])
```

Now find the eigenvalues for the first order system. For what values of b *do solutions oscillate? Can you interpret your results "physically"?*