Chapter 3

Metric spaces

3.1 Definition

Definition 3.1. *Let X be a set. A function $d\colon X \times X \to [0, \infty)$ is a **metric** on X if d is*

- *definite, meaning that for all $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$,*
- *symmetric, meaning that $d(x, y) = d(y, x)$ for all $x, y \in X$, and*
- *satisfies the triangle inequality*

$$d(x, y) \leq d(x, z) + d(z, y)$$

  for all $x, y, z \in X$.

*If d is a metric on X then the pair (X, d) is called a **metric space**.*

Example 3.2.

(a) *The function $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$. We refer to this as the **standard metric** on $\mathbb{R}$.*

(b) *In the exercises, you show that*

$$d_2((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

*is a metric on $\mathbb{R}^2$. We refer to this as the **standard metric** on $\mathbb{R}^2$.*

(c) *Let X be any set. The function defined by*

$$d_0(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

is a metric. This metric is called the ***discrete metric*** *on X.*

(d) *Let X be the set of all lines in $\mathbb{R}^3$ passing through the origin. Define a metric by declaring the distance between two lines to be the radian measure of the angle between them. This metric space is called the* ***projective plane****.*

Definition 3.3. *Suppose (X, d) is a metric space. We say that a sequence $\{x_k\}_{k=k_0}^{\infty} \subset X$* ***converges*** *to x_*, and write $x_k \to x_*$, if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$n \geq N \implies d(x_n, x_*) < \varepsilon.$$

Exercise 3.1. Show that

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

is a metric on $\mathbb{R}$.

Exercise 3.2. Show that the following functions are metrics on $\mathbb{R}^2$:

(a) $d_\infty((x_1,x_2),(y_1,y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|)$

(b) $d_1((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$

(c) $d_2((x_1,x_2),(y_1,y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$

Exercise 3.3. For each of the metrics in the previous exercise, make a sketch of the set of points $(x,y) \in \mathbb{R}^2$ such that

$$d_*((x,y),(0,0)) = 1.$$

(Here the star stands for any of $\infty, 1, 2$.)

Exercise 3.4. Let $\{(x_k, y_k)\}$ be a sequence in $\mathbb{R}^2$. Show that

$$(x_k, y_k) \to (x_*, y_*)$$

with respect to the standard metric on $\mathbb{R}^2$ if and only if both

$$x_k \to x_* \quad \text{and} \quad y_k \to y_*$$

with respect to the standard metric on $\mathbb{R}$.

3.2 Open sets

For the purposes of this section we assume that (X, d) is a metric space.

Definition 3.4. *Let $p \in X$ and $r \in (0, \infty)$. The **open ball of radius r** is defined to be the set*

$$B_r(p) = \{q \in X \mid d(p, q) < r\}.$$

Example 3.5. *Let d_1 and d_2 be as in Exercise 3.2.*

(a) *In $(\mathbb{R}^2, d_2)$ the open ball $B_r(p)$ consists of all points inside the circle of radius r centered at p.*

(b) *What are the open balls in $(\mathbb{R}^2, d_\infty)$? What about $(\mathbb{R}^2, d_1)$?*

Definition 3.6. *Let $Y \subset X$.*

(a) *A point $p \in Y$ is an **interior point** if there exists $r > 0$ such that $B_r(p) \subset Y$.*

(b) *We say that Y is **open in X** if each element of Y is an interior point.*

Example 3.7. *Let $X = [0, 1]$ and $d(x, y) = |x - y|$. The following sets are open in X: $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{2}, 1)$, $(\frac{1}{2}, 1]$, $[0, 1]$.*

Proposition 3.8. *In a metric space (X, d) the whole set X and the empty set $\emptyset$ are both open.*

Proposition 3.9.

(a) *The intersection of any finite collection of open sets is an open set.*

(b) *The union of any collection (not necessarily countable!) of open sets is again an open set.*

Example 3.10. (a) *For $n \in \mathbb{N}$ let $Y_n = (1/n, 2)$. Note that each Y_n is open. Furthermore*

$$\bigcup_{n \in \mathbb{N}} Y_n = (0, 2)$$

is also open.

(b) *For each $n \in \mathbb{N}$ let $U_n = (-1/n, 1/n)$. Note that each U_n is open. For each $N \in \mathbb{N}$ we have that*

$$\bigcap_{n=1}^{N} U_n = (-1/N, 1/N)$$

is open. But

$$\bigcap_{n \in \mathbb{N}} U_n = \{0\}$$

is not open.

Exercise 3.5. Suppose set X is equipped with the discrete metric. What are the open sets?

Exercise 3.6. Consider the metric space where

$$X = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$$

and d is the standard metric on $\mathbb{R}^2$.

(a) Draw a picture of $B_1((1, 1))$.

(b) Draw a picture of $B_{\sqrt{5}/2}((0, 0))$.

(c) Give a general description of the open sets in this metric space.

Exercise 3.7. Consider the set $Y = \mathbb{R} \setminus \{1/n \mid n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$ equipped with the standard metric. Is Y open?

Exercise 3.8.

(a) Describe the open balls of $\mathbb{R}^2$ determined by the metric d_1.

(b) Prove that a subset of $\mathbb{R}^2$ is open with respect to d_1 if and only if it is open with respect to d_2.

Exercise 3.9. Prove that the intersection of a finite number of open sets is an open set.

Exercise 3.10 (Challenge!)**.** Show that $x_k \to x_*$ in metric space (X, d) if and only if for every open set $U \ni x_*$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies x_n \in U.$$

3.3 Closed sets

Throughout this section (X, d) is a metric space.

Definition 3.11. *Let $Y \subset X$.*

(a) *A point $p \in X$ is a **limit point of Y** if there exists a sequence $\{x_k\} \subset Y \setminus \{p\}$ such that $x_k \to p$.*

(b) *A point $p \in Y$ that is not a limit point of Y is called an **isolated point of Y**.*

Example 3.12. *Consider the subset Y of $\mathbb{R}$ with the standard metric given by $Y = (-\infty, 0] \cup \{1\} \cup [2, 3)$. The limit points of Y are $(-\infty, 0] \cup [2, 3]$. So we see that not all elements of Y are limit points, and not all limit points of Y are contained in Y. Furthermore, Y contains has an isolated point at 1.*

Example 3.13. *Consider the set $K \subset \mathbb{R}$ given by $K = \{1/n \mid n \in \mathbb{N}\}$. Every point of K is an isolated point. The point 0 is a limit point of K, though $0 \notin K$.*

Definition 3.14. ***Y is closed in X** if Y contains all its limit points.*

Example 3.15. *Consider the metric space where $X = [0, 1]$ and d is the standard metric on $\mathbb{R}$.*

(a) *The following sets are closed in X: $[0, 1/2]$, $[1/3, 1/2]$, $\{1/5\}$.*

(b) *The following sets are open in X: $[0, 1/2)$, $(1/3, 1/2)$.*

(c) *The following sets are neither open nor closed in X: $(0, 1/2]$, $[1/3, 1/2)$.*

(d) *Finally, the sets $[0, 1]$ and $\emptyset$ is both open and closed in X.*

Proposition 3.16. *A set Y is closed in X if and only if $X \setminus Y$ is open.*

Proposition 3.17.

(a) *The sets X and $\emptyset$ are closed in X.*

(b) *The union of a finite number of sets closed in X is closed in X.*

(c) *The intersection of an arbitrary number of sets closed in X is closed.*

Example 3.18. *Consider the sets $V_n = \left[\frac{1-n}{n}, \frac{n-1}{n}\right]$, where $n \in \mathbb{N}$. The union of a finite number of the V_n sets is still closed, but $\bigcup_{n \in \mathbb{N}} V_n = (-1, 1)$, which is not closed.*

Definition 3.19. *A set $Y \subset X$ is **connected** if the only subsets of Y that are open in Y are Y and $\emptyset$.*

Remark 3.20. *The following equivalent characterization of connected-ness is useful: A set $Y \subset X$ is not connected precisely when $Y = Y_1 \cup Y_2$ with $Y_1 \cap Y_2 = \emptyset$, both $Y_1, Y_2 \neq \emptyset$, and both Y_1, Y_2 open in Y. (Note that both Y_1, Y_2 open in Y means that both Y_1, Y_2 are closed in Y as well.)*

Exercise 3.11. Find all limit points of the following subsets of $\mathbb{R}$ (where we use the standard metric).

(a) $\{1/(1+x^2) \mid x \in \mathbb{R}\}$

(b) $(0,1) \setminus \{1/n \mid n \in \mathbb{N}\}$

(c) $\mathbb{Q}$

Exercise 3.12. Give an example of a nonempty subset of $\mathbb{R}^2$ (with the standard metric) that is

(a) closed, but not open;

(b) open, but not closed;

(c) both open and closed;

(d) neither open nor closed.

Draw pictures of your sets.

Exercise 3.13. Let $Y \subset \mathbb{R}^2$, equipped with the standard metric. Show that if Y is open then every interior point of Y is also a limit point.

Exercise 3.14. Prove Proposition 3.17.

Exercise 3.15. Let (X,d) be a metric space and suppose $Y \subset X$. We define the ***closure*** of Y in X, which we give the symbol $\bar{Y}$, by

$$\bar{Y} = \bigcap F,$$

where the intersection is taken over the set of all closed sets F such that $Y \subset F$. In other words, $\bar{Y}$ is the smallest closed set containing Y.

Show that $x \in \bar{Y}$ if and only if $B_\varepsilon(x) \cap Y \neq \emptyset$ for all $\varepsilon > 0$.

3.4 Continuous functions on metric spaces

Definition 3.21. *Suppose X, Y are metric spaces.*

(a) *A function $f\colon X \to Y$ is* ***continuous at $x_* \in X$*** *if for any sequence $\{x_k\} \subset X$ such that $x_k \to x_*$ in X we have $f(x_k) \to f(x_*)$ in Y.*

(b) *A function $f\colon X \to Y$ is* ***continuous*** *if it is continuous at every $x_* \in X$.*

Example 3.22.

(a) *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous.*

(b) *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

is continuous everywhere except at $x = 0$.

(c) *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous only at $x = 0$.

Theorem 3.23. *Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f\colon X \to Y$. Then the following are equivalent.*

(a) *f is continuous.*

(b) *For each $x_* \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ (possibly depending on both x_* and ε) such that*

$$d_X(x, x_*) < \delta \implies d_Y(f(x), f(x_*)) < \varepsilon.$$

(c) *For each open set $U \subset Y$ the preimage $f^{-1}(U)$ is open in X.*

(d) *For each closed set $V \subset Y$ the preimage $f^{-1}(V)$ is closed in X.*

Remark 3.24. *Property* (b) *in the previous theorem is called the "ε–δ" definition of continuity, and is equivalent to the following* (b′) *For each* $x_* \in X$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *(possibly depending on both* x_* *and* ε*) such that*

$$f(B_\delta(x_*)) \subset B_\varepsilon(f(x_*)).$$

Example 3.25. *If* $f\colon X \to Y$ *is a continuous function, then the preimages of open sets are open. However, the images of open sets are not necessarily open. For example, consider the function* $f\colon \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^2$*. The set* $U = (-1, 1)$ *is open, but its image is not.*

The proofs of the next two theorems show the utility of being able to move between equivalent definitions of continuity.

Theorem 3.26. *The composition of continuous functions is continuous.*

Theorem 3.27. *The continuous image of a connected set is connected.*

Definition 3.28. *Suppose* (X, d_X) *and* (Y, d_Y) *are metric spaces. A function* $f\colon X \to Y$ *is* ***uniformly continuous*** *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $x_1, x_2 \in X$ *we have*

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Exercise 3.16. Prove that the following functions are continuous using the ε–δ definition:

(a) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

(b) $f\colon (0, \infty) \to \mathbb{R}$ given by $f(x) = \dfrac{1}{x}$

(c) $f\colon [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$

Exercise 3.17. Prove that the following functions are uniformly continuous.

(a) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{1}{1 + x^2}$

(b) $f\colon [1, \infty) \to \mathbb{R}$ given by $f(x) = \dfrac{1}{\sqrt{x}}$

(c) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{x}$

Exercise 3.18. Prove that the following functions are not uniformly continuous:

(a) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

(b) $f\colon (0, \infty) \to \mathbb{R}$ given by $f(x) = \dfrac{1}{\sqrt{x}}$

Exercise 3.19. Find a continuous function $f\colon X \to Y$ and a closed set $V \subset X$ such that $f(V)$ is not closed in Y.

Exercise 3.20. Consider the function $f\colon [0,1] \cup (2,3] \to [0,2]$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1], \\ x - 1 & \text{if } x \in (2,3]. \end{cases}$$

(a) Draw a sketch of the graph of f.

(b) Prove that f is continuous.

(c) Explain why f has an inverse function f^{-1}. Draw a sketch of the graph of f^{-1}.

(d) Prove that f^{-1} is not continuous.

(e) What lesson does this example teach us?

Exercise 3.21. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x,y)$.

(a) Show that f is continuous when we use the metric d_1 in the domain and the metric d_2 in the codomain.

(b) Show that f is continuous when we use the metric d_∞ in the domain and the metric d_1 in the codomain.

(c) Show that f is continuous when we use the metric d_0 in the domain and the metric d_∞ in the codomain.

(d) Show that f is *not* continuous when we use the metric d_2 in the domain and the metric d_0 in the codomain.

Exercise 3.22 (Challenge)**.** Suppose that both d and d' are metrics on set X. We say that d and d' are ***equivalent*** if a set $U \subset X$ is open with respect to d if and only if it is open with respect to d'.

(a) Which of the metrics d_0, d_1, d_∞ on $\mathbb{R}^2$ are equivalent to the standard metric d_2?

(b) Give an equivalent (pun intended) definition of (X, d) being equivalent to (X, d') that uses continuous functions.

3.5 Compact sets

Definition 3.29. *Let (X, d) be a metric space and let $Y \subset X$. We say that Y is **compact** if for every sequence $\{y_k\} \subset Y$ there exists a subsequence $\{y_{k_j}\}$ that converges to a point $y \in Y$.*

Definition 3.30.

(a) *Let (X, d) be a metric space. A set $Y \subset X$ is **bounded** if there exists $p \in X$ and $M > 0$ such that $d(p, y) < R$ for all $y \in Y$.*

(b) *Suppose $f : X \to Y$ is a function of metric spaces. We say that the function f is **bounded** if $f(X)$ is a bounded subset of Y.*

Theorem 3.31 (Heine-Borel Theorem)**.** *Suppose $K \subset \mathbb{R}$. The following are equivalent:*

(a) *K is closed and bounded.*

(b) *K is compact.*

Remark 3.32. *The Heine-Borel theorem extends to subsets of $\mathbb{R}^n$.*

Theorem 3.33. *Suppose $f : X \to Y$ is a continuous function between metric spaces. If X is compact, then $f(X)$ is compact.*

Remark 3.34. *If $Y \subset X$ is bounded, then in fact for each $q \in Y$ there exists $M_q > 0$ such that $d(q, y) < M_q$ for all $y \in Y$.*

Theorem 3.35 (Extreme value theorem)**.** *Suppose that $f : X \to \mathbb{R}$ is continuous and bounded, and that X is compact. Then there exists $x_* \in X$ such that $f(x_*) \geq f(x)$ for all $x \in X$.*

→ Needed: continuous functions with compact domain are uniformly continuous

Theorem 3.36 (Challenge!)**.** *Let (X, d) be a metric space. The following properties are equivalent:*

(a) *X is compact.*

(b) *If U_α is a (not necessarily countable) collection of open subsets of X such that $\cup U_\alpha = X$, then there exists a finite number of the sets $U_{\alpha_1}, \ldots, U_{\alpha_N}$ such that $U_{\alpha_1} \cup \cdots \cup U_{\alpha_N} = X$.*

The second condition, that any "open covering" of X has a finite "subcover," is the topological definition of compactness.

3.6 Supremums and infimums

Definition 3.37. *Suppose $X \subset \mathbb{R}$.*

(a) *We say that M is a **least upper bound** of X if*

  (i) *$x \leq M$ for all $x \in X$, and*

  (ii) *for all $y < M$ there exists $x \in X$ with $y < x$.*

(b) *We say that m is a **greatest lower bound** of X if*

  (i) *$x \geq m$ for all $x \in X$, and*

  (ii) *for all $y > m$ there exists $x \in X$ with $y > x$.*

Theorem 3.38.

(a) *If $X \subset \mathbb{R}$ is bounded from above and is nonempty, then X has a unique least upper bound, which we call the **supremum** of X and denote $\sup X$.*

(b) *If $X \subset \mathbb{R}$ is bounded from below and is nonempty, then X has a unique greatest lower bound, which we call the **infimum** of X and denote $\inf X$.*

Remark 3.39. *If X is not bounded from above, then we write $\sup X = \infty$, etc.*

Example 3.40.

- $\sup[0, 1) = 1$.
- $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$.
- $\sup\{x \in \mathbb{Q} : x^2 < 2\} = \sqrt{2}$

Theorem 3.41. *If $X \subset \mathbb{R}$ is compact, then $\sup X \in X$ and $\inf X \in X$.*

Definition 3.42. *Suppose $f : X \to \mathbb{R}$ is a continuous function of metric spaces. The **supremum of f** is defined by*

$$\sup_X f = \sup\{f(x) : x \in X\}.$$

The infimum of f is defined in a similar manner.

Example 3.43.

(a) *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be given by* $f(x) = \frac{1}{1+x^2}$. *Then*

$$\sup f = 1, \qquad \inf f = 0.$$

(b) *Let* $f\colon (0,\infty) \to \mathbb{R}$ *be given by* $f(x) = \frac{1}{x}$. *Then*

$$\sup f = \infty, \qquad \inf f = 0.$$

Definition 3.44. *Let X be a metric space. We denote by $C(X)$ the collection of continuous functions $X \to \mathbb{R}$. We define the **supremum norm** of a function $f \in C(X)$ by*

$$\|f\| = \sup_X |f|.$$

Let $C_b(X)$ be the functions in $C(X)$ for which the supremum norm is finite.

Example 3.45. *Suppose $X = (0,1)$.*

(a) *Let $f \in C(X)$ be defined by $f(x) = x^2$. Then $\|f\| = 1$ and $f \in C_b(X)$.*

(b) *Let $j \in C(X)$ be defined by $j(x) = \frac{x}{x+1}$. Then $\|j\| = \frac{1}{2}$ and $j \in C_b(X)$.*

(c) *Let $g \in C(X)$ be defined by $g(x) = \frac{1}{x}$. Then $\|g\| = \infty$ and $g \notin C_b(X)$.*

Theorem 3.46. *$d(f,g) = \|f - g\|$ makes $C_b(X)$ a metric space.*

Remark 3.47. *The statement that $f_n \to f$ in $C_b(X)$ is stronger than the statement that $f_n(x) \to f(x)$ for all $x \in X$. To see this, consider the functions $f_n\colon [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n$. For each fixed $x \in [0,1]$ we have*

$\rightarrow$ **need to state explicitly that convergence in norm implies pointwise convergence**

$$f_n(x) \to f_\infty(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

But $\|f_n - f_\infty\| \geq \frac{1}{2}$ for each $n \in \mathbb{N}$.

Exercise 3.23. Suppose $X = \{1,2\}$ is equipped with the discrete metric. Describe $C(X)$.

Exercise 3.24. Suppose that X is compact. Show that $C(X) = C_b(X)$.

3.7 Complete metric spaces

Definition 3.48. *A metric space (X, d) is **complete** if all Cauchy sequences in X converge to an element of X.*

Example 3.49.

- $\mathbb{R}$ *is complete (with respect to the standard metric)*
- $\mathbb{R}^n$ *is complete (with respect to the standard metric)*
- $\mathbb{Q}$ *is* not *complete (with respect to the standard metric)*
- $(0, 1)$ *is* not *complete (with respect to the standard metric)*

Theorem 3.50. *Suppose (X, d) is a complete metric space and suppose that $Y \subset X$ is closed. Then (Y, d) is a complete metric space.*

Example 3.51. *The closed unit ball in $\mathbb{R}^n$ is complete (with respect to the standard metric).*

Theorem 3.52. *If X is a metric space then $C_b(X)$ is complete metric space.*

The proof of the theorem consists of the following lemmas.

Lemma 3.53. *Suppose $\{f_n\} \subset C_b(X)$ is Cauchy. Then there exists a function $f: X \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbb{X}$.*

Hint: for each $x \in X$ the sequence $\{f_n(x)\}$ is Cauchy.

Lemma 3.54. *Suppose $\{f_n\} \subset C_b(X)$ is Cauchy; let $f: X \to \mathbb{R}$ be the function such that $f_n(x) \to f(x)$ for each $x \in X$. For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$m \geq N \implies \|f_m - f\| < \varepsilon.$$

Hint: let $x \in X$ and $m \in \mathbb{N}$, then $|f_m(x) - f_n(x)| \to |f_m(x) - f(x)|$.

Lemma 3.55. *Suppose $\{f_n\} \subset C_b(X)$ is Cauchy; let $f: X \to \mathbb{R}$ be the function such that $f_n(x) \to f(x)$ for each $x \in X$. Then f is continuous.*

Hint: use the continuity of f_m and an $\varepsilon/3$ argument.

3.8 Applications of completeness

Definition 3.56. *Suppose $f\colon X \to X$. An element $x \in X$ is a **fixed point** of f if $f(x) = x$.*

Proposition 3.57. *Suppose $f\colon [0,1] \to [0,1]$ is continuous. Then f has at least one fixed point.*

Definition 3.58. *Let (X, d) be a metric space. A function $f\colon X \to X$ is a **contraction mapping** if there exists $\alpha \in (0,1)$ such that*

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$.

Theorem 3.59. *Suppose X is a complete metric space and $f\colon X \to X$ is a contraction mapping. Then f has a unique fixed point.*

Exercise 3.25. Proof Proposition 3.57 using the intermediate value theorem.

Exercise 3.26. Prove Theorem 3.59 using the following steps:

(a) Let $x_0 \in X$. Recursively define a sequence $\{x_k\} \subset X$ by $x_k = f(x_{k-1})$.

(b) Show that for any $n, m \in \mathbb{N}$ we have

$$d(x_n, x_m) \leq \sum_{k=n+1}^{m} d(x_k, x_{k-1}).$$

(c) Show that for any $k \in \mathbb{N}$ we have $d(x_k, x_{k+1}) \leq \alpha^k d(x_0, x_1)$.

Exercise 3.27.

$$\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Quiz: Topological Equivalence

- This quiz is to be completed independently, meaning that you are not to consult any source other than Paul, the course handouts, and your notes. In particular, no internet, no other books, no discussing with others (classmates, friends, professors, etc.).
- Please write your responses on blank copy paper, writing only on one side, and attach to this page.

Definition 3.60. *Suppose (X, d_X) and (Y, d_Y) are metric spaces. We say that X and Y are* ***topologically equivalent*** *if there exists a bijection $\varphi\colon X \to Y$ that is continuous, and such that $\varphi^{-1}\colon Y \to X$ is also continuous.*

Problem 1 (Examples)**.**

(a) Construct a function $\varphi\colon [0,1) \to [0,\infty)$ in order to prove that $[0,1)$ and $[0,\infty)$ are equivalent (when using the standard metric on both spaces).

(b) Show that the unit open ball $B_1(0,0)$ is topologically equivalent to $\mathbb{R}^2$, where we use the standard metric on both.

(c) Let the unit square $\square \subset \mathbb{R}^2$ be defined by

$$\square = \{(x,y) : |x| = 1, |y| \leq 1\} \cup \{(x,y)\colon |x| \leq 1, |y| = 1\}.$$

Let the unit circle $S^1 \subset \mathbb{R}^2$ be defined by

$$S^1 = \{(x,y) : x^2 + y^2 = 1\}.$$

We use the standard metric d_2 on both $\square$ and S^1.

Show that $\square$ and S^1 are topologically equivalent.

Problem 2 (Basic theory)**.**

(a) The word "topological" can be interpreted to mean "having to do with open sets." What does topological equivalence have to do with open sets?

(b) Show that being topologically equivalent is an equivalence relation on the set of all metric spaces.

Problem 3. Suppose that (X, d_X) and (Y, d_Y) are topologically equivalent.

(a) Show that X is compact if and only if Y is compact.

(b) Show that X is connected if and only if Y is connected[1].

Thus compactness and connectedness are "topological properties."

Problem 4 (Non-examples)**.** Show that X and Y are not topologically equivalent in the following situations.

(a) $X = [0, 1)$ and $Y = [0, 1]$, where we use the standard metric for both.

(b) X and Y are the following subsets of $\mathbb{R}^2$ and use the standard metric:

$$X = \{(x, y) : x^2 + y^2 = 1\},$$
$$Y = \{(x, y) : x^2 - y^2 = 1\}.$$

(c) $X = [0, 1]$ with the standard metric, $Y = [0, 1]$ with the discrete metric.

Problem 5. Assume for the purposes of this problem that all of the properties of cosine and sine (and pi) have been established. Consider the function $f\colon [0, 2\pi) \to S^1$, where $S^1 = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$, given by

$$f(x) = (\cos(x), \sin(x)).$$

Give an "intuitive" (non-proof) answer to the following.

- Does f show that $[0, 2\pi)$ is topologically equivalent to S^1? If so, explain why. If not, what goes wrong?

[1]Recall: X is connected means that the only subsets of X that are both open and closed are X and $\emptyset$.