Chapter 2

The real numbers

Main theorem that there exists a set $\mathbb{R}$, called the ***real numbers***, that

- is an ordered field,
- satisfies the Archimedean property, and
- in which bisection search is successful.

Furthermore, this set is unique "in some appropriate sense." From now on, we assume we are working with this set. At the end of the course we'll address the existence and uniqueness.

2.1 Integer roots

Proposition 2.1. *Let $p \in \mathbb{N}$ and $y > 0$. Then there exists $x > 0$ such that $x^p = y$.*

2.2 Series of constants

Definition 2.2 (Convergence of infinite series)**.** *Suppose $\{a_k\}_{k=k_0}^{\infty}$ is a sequence of real numbers. We say that the corresponding infinite series converges if the sequence of partial sums*

$$S_n = \sum_{k=k_0}^{n} a_k$$

is a convergent sequence. If $S_n \to S$ then we write

$$\sum_{k=k_0}^{\infty} a_k = S.$$

If S_n is not a convergent sequence, then we say that the series diverges.

Remark 2.3. *Since the exact value of k_0 is not relevant to whether the series converges, we often simply write $\sum a_k$ converges.*

Example 2.4. *From earlier homework we have seen that*

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

converges, and that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

Proposition 2.5 (Cauchy criterion for convergent series)**.** *An infinite series $\sum_{k=k_0}^{\infty} a_k$ converges if and only if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$m \geq n \geq N \implies \left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

Corollary 2.6. *If $\sum a_k$ converges then $a_k \to 0$.*

Example 2.7. *The series $\sum \frac{k}{k+1}$ does not converge.*

Proposition 2.8 (Comparison test for series)**.** *Suppose that a_k and b_k are sequences with $|a_k| \leq b_k$ for all $k \in \mathbb{N}$. Suppose also that $\sum b_k$ converges. Then $\sum a_k$ converges.*

Example 2.9. *Consider the series*

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k3^k}.$$

Note that for $k \geq 1$ we have

$$\left| \frac{(-2)^k}{k3^k} \right| \leq \left(\frac{2}{3} \right)^k.$$

Since $\sum \left(\frac{2}{3}\right)^k$ *converges, we have that*

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k3^k}$$

converges.

Proposition 2.10 (Ratio test for series)**.** *Suppose* a_k *is a sequence.*

(a) *If there exists* $K \in \mathbb{N}$ *and* $r < 1$ *such that*

$$k \geq K \implies \left|\frac{a_{k+1}}{a_k}\right| \leq r$$

Then $\sum a_k$ *converges.*

(b) *If there exists* $K \in \mathbb{N}$ *and* $r > 1$ *such that*

$$k \geq K \implies \left|\frac{a_{k+1}}{a_k}\right| \geq r$$

Then $\sum a_k$ *diverges.*

Example 2.11. *Consider the series*

$$\sum_{k=0}^{\infty} \frac{1}{k!}.$$

It is easy to see from the ratio test that the series converges. We denote the limit of this series by e*.*

Example 2.12. *Consider the series*

$$\sum_{k=1}^{\infty} \frac{x^k}{k},$$

where x *is some real number. By the ratio test we see that the series converges when* $|x| < 1$*.*

Exercise 2.1.

(a) Proof Proposition 2.5 by applying the Cauchy criterion for sequences.

(b) Prove that if $\sum a_k < \infty$ then $a_k \to 0$.

Exercise 2.2. Under what conditions does the series $\sum(a_k + b_k)$ converge? What about $\sum(\alpha a_k)$, where α is a constant?

Exercise 2.3. For what values of x do the following series converge? Provide justification.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{x^{2k}}$

(b) $\displaystyle\sum_{k=0}^{\infty} (k+1)(-x)^k$

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-)^k x^{2k}}{(2k)!}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x-1}{x}\right)^k$

Exercise 2.4. Do the following sequences converge? Provide justification.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k 5^{2k+1}}{(2k+1)!}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{2^k}$

(c) (Challenge!) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

(d) (Challenge!) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

2.3 Monotone sequences

Definition 2.13 (Monotone sequence)**.**

(a) *A sequence* $\{x_k\}$ *is* ***monotone increasing*** *if* $x_k \leq x_{k+1}$ *for all* $k \in \mathbb{N}$. *We say that* $\{x_k\}$ *is* ***strictly monotone increasing*** *if* $x_k < x_{k+1}$ *for all* $k \in \mathbb{N}$.

(b) *A sequence* $\{x_k\}$ *is* ***monotone decreasing*** *if* $x_k \geq x_{k+1}$ *for all* $k \in \mathbb{N}$. *We say that* $\{x_k\}$ *is* ***strictly monotone decreasing*** *if* $x_k > x_{k+1}$ *for all* $k \in \mathbb{N}$.

If a sequence is either monotone increasing or monotone decreasing, then we say that it is ***monotone***

Proposition 2.14 (Convergence criterion for monotone sequences)**.** *A monotone sequence converges if and only if it is bounded.*

Example 2.15. *Many example of monotone sequences are series with positive terms. For example, consider the series*

$$\sum_{k=1}^{\infty} \frac{1}{k2^k}.$$

The partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{k2^k}$$

for a monotone increasing sequence. Note that

$$S_n = \sum_{k=1}^{n} \frac{1}{k}\frac{1}{2^k} \leq \sum_{k=1}^{n} \frac{1}{2^k} \leq \frac{1}{1-\frac{1}{2}}.$$

Thus S_n *is bounded. Since* S_b *is bounded and monotone, the series must converge.*

Corollary 2.16. *If* $a_k \geq 0$ *for all* k *then* $\sum a_k$ *converges if and only if partial sums are bounded.*

Proposition 2.17 (Alternating series test)**.** *Suppose* b_k *is a monotone decreasing sequence such that* $b_k \to 0$. *Then*

$$\sum_{k=k_0}^{\infty} (-1)^k b_k$$

converges.

Example 2.18. *Consider the series*

$$\sum_{k=0}^{\infty} \frac{4(-1)^k}{2k+1},$$

We denote the limit of this series as π.

Exercise 2.5 (p test for series)**.** In this exercise, you show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges. Since the sequence of partial sums is monotone, the plan is to show that they are bounded.

(a) Consider the partial sum $S_{15} = \sum_{k=1}^{15} \frac{1}{k^2}$. We organize the partial sum as follows:

$$\begin{aligned} S_{15} = & \frac{1}{1^2} \\ & + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) \\ & + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) \\ & + \left(\frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2}\right) \end{aligned}$$

Notice that

$$\left(\frac{1}{2^2} + \frac{1}{3^2}\right) \leq \left(\frac{1}{2^2} + \frac{1}{2^2}\right) = 2\frac{1}{2^2} = \frac{1}{2}.$$

Show that

$$\left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) \leq \frac{1}{4}$$

$$\left(\frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2}\right) \leq \frac{1}{8}$$

Conclude that

$$S_{15} \leq \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3}.$$

(b) Obtain the analogous estimate for S_{31}.

(c) Let $N \in \mathbb{N}$. Find a way to break up the partial sum S_{2^N-1} in to collections of terms that can be bounded using a similar procedure. Use this to prove (by induction on N) that

$$S_{2^N-1} \le \sum_{l=0}^{N-1} \frac{1}{2^l}.$$

(d) Explain why

$$\sum_{l=0}^{N-1} \frac{1}{2^l} \le 2$$

for all N.

(e) Prove that S_n is bounded, and thus that the series converges.

(f) (Challenge!) Show how to generalize this idea to show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges when $p > 1$.

(g) (More challenge!!) Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

diverges when $p \le 1$.

2.4 Activity: the number e.

This activity is based on the section "The Number e" in Rudin's book Principles of Mathematical Analysis*.*

(a) Recall that we define e to be the limit of the sequence S_m given by

$$S_m = \sum_{k=0}^{m} \frac{1}{k!}.$$

(b) Define the sequence T_n by

$$T_n = \left(1 + \frac{1}{n}\right)^n.$$

Use the binomial theorem to show that

$$T_n = \sum_{k=0}^{n} \frac{1}{n^k}\binom{n}{k} = \sum_{k=0}^{n} \frac{1}{k!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right).$$

(c) Suppose that k, n are integers with $0 \le k \le n$. Show that

$$\frac{1}{n^k}\binom{n}{k} \le \frac{1}{k!}.$$

Use this to show that $T_n \le e$ for all $n \in \mathbb{N}$.

(d) Suppose that k, n are integers with $0 \le k \le n$. Show that

$$\frac{1}{n^k}\binom{n}{k} \le \frac{1}{(n+1)^k}\binom{n+1}{k}$$

Use the previous steps to conclude that T_n is monotone increasing, and therefore has a limit. Let T be the limit of the sequence T_n.

(e) Let $m \in \mathbb{N}$. Show that for $n > m$ we have

$$T_n \ge \sum_{k=0}^{m} \frac{1}{k!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m}{n}\right).$$

Use this to show that $T \ge S_m$ for all $m \in \mathbb{N}$.

(f) Deduce that $T = e$ and thus

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

2.5 Subsequences

- Definition of subsequence
- HW: If a sequence converges, all subsequences converge (to that limit)
- Prop: every sequence has a monotone subsequence
- Cor: every bounded sequence has a convergent subsequence
- Cor: (Bolzano-Weierstrass) Any sequence contained in $[a, b]$ has a subsequence that converges in that interval.