Chapter 1

Numbers

One foundational issue in analysis is to clarify precisely what properties of numbers must be assumed in order to "do calculus" and to understand the existence of numbers that satisfying those properties. The goal of this chapter is to address the first part of that issue: Throughout the next several sections we motivate a list of properties that we want/need the real numbers to have. While compiling this list we take care to observe which properties are fundamental (in the sense that they need to be taken as axioms) and which are derived (in that they are consequences of the axioms). Once we have complies our list, we proceed with the task of "making calculus precise," all the while simply assuming that the real numbers have these properties. Finally, in the last chapter of the course, we turn to the question of existence and discuss ways to construct models of the real numbers having the desired properties.

1.1 Ordered fields

The most basic properties we assume about the real numbers are described by the following definition.

Definition 1.1 (Field)**.** *A set $\mathbb{F}$ is a* ***field*** *of there are operations of* ***addition****, denoted $+$, and* ***multiplication****, denoted $\cdot$, taking*

$$\mathbb{F} \times \mathbb{F} \to \mathbb{F},$$

and such that

- *addition is commutative: $x + y = y + x$ for all $x, y \in \mathbb{F}$;*

- *addition is associative:* $(x + y) + z = x + (y + z)$ *for all* $x, y, z \in \mathbb{F}$*;*
- *there exists an* ***additive identity*** $0 \in \mathbb{F}$ *such that* $0 + x = x$ *for all* $x \in \mathbb{F}$*;*
- *for every* $x \in \mathbb{F}$ *there exists an* ***additive inverse*** $-x \in \mathbb{F}$ *such that* $x + (-x) = 0$

and

- *multiplication is commutative:* $x \cdot y = y \cdot x$ *for all* $x, y \in \mathbb{F}$*;*
- *multiplication is associative:* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *for all* $x, y, z \in \mathbb{F}$*;*
- *there exists a* ***multiplicative identity*** $1 \in \mathbb{F} \setminus \{0\}$ *such that* $1 \cdot x = x$ *for all* $x \in \mathbb{F}$*;*
- *for every* $x \in \mathbb{F} \setminus \{0\}$ *there exists a* ***multiplicative inverse*** $x^{-1} \in \mathbb{F}$ *such that* $x \cdot x^{-1} = 1$*;*

and

- *the distributive law* $x \cdot (y + z) = x \cdot y + x \cdot z$ *holds for all* $x, y, z \in \mathbb{F}$*.*

Remark 1.2. *As the following example illustrates, there are many different fields, not just the real numbers. The importance of these examples is that if a property of the real numbers is a consequence of the real numbers being a field (that is, if the property is able to be deduced from the definition of a field), then all examples of fields must have that property. Thus we can use examples of fields (and of specific types of fields that are defined below) in order to "test out" what features of the real numbers might possibly be deduced from the properties being assumed.*

Example 1.3.

(a) *If* p *is a prime number then* $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$*, with the usual arithmetic done modulo* p*, forms a field.*

(b) *The rational numbers* $\mathbb{Q}$ *form a field.*

An additional property that we need is a concept of "less than."

Definition 1.4 (Ordered field)**.** *An* ***ordered field*** *is a field* $\mathbb{F}$ *together with a relation* $<$ *such that*

- *for each* $x, y \in \mathbb{F}$ *precisely one of the following holds:*

$$x < y, \quad x = y, \quad y < x;$$

- *if* $x, y, z \in \mathbb{F}$ *with* $x < y$ *and* $y < z$*, then* $x < z$*;*
- *if* $x, y, z \in \mathbb{F}$ *with* $x < y$*, then* $x + z < y + z$*;*
- *if* $x, y \in \mathbb{F}$ *with* $0 < x$ *and* $0 < y$*, then* $0 < x \cdot y$*.*

Many of the features of real numbers that we need are immediate consequences of being an ordered field.

Proposition 1.5. *Suppose* $\mathbb{F}$ *is an ordered field. The following basic properties of addition hold:*

- *Additive cancellation: if* $x + y = x + z$ *then* $y = z$*.*
- *Uniqueness of zero: if* $x + y = x$ *then* $y = 0$*.*
- *Uniqueness of additive inverses: if* $x + y = 0$ *then* $y = -x$*.*
- *Involution of additive inverse:* $-(-x) = x$

The following basic properties of multiplication hold:

- *Multiplicative cancellation: if* $x \cdot y = x \cdot z$ *and* $x \neq 0$ *then* $y = z$*.*
- *Uniqueness of* 1*: if* $x \cdot y = x$ *and* $x \neq 0$ *then* $y = 1$*.*
- *Uniqueness of multiplicative inverses: if* $x \cdot y = 1$ *then* $y = x^{-1}$*.*
- *Involution of multiplicative inverse: if* $x \neq 0$ *then* $(x^{-1})^{-1} = x$

The following additional properties of multiplication hold:

(a) $0 \cdot x = 0$.

(b) *If* $x, y \neq 0$ *then* $x \cdot y \neq 0$*.*

(c) $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$.

(d) $(-x) \cdot (-y) = x \cdot y$.

Finally, the following properties of order hold:

(e) *If* $x > 0$ *then* $-x < 0$*; if* $y < 0$ *then* $-y > 0$*.*

(f) *If* $x > 0$ *and* $a < b$ *then* $xa < ab$*; if* $y < 0$ *and* $a < b$ *then* $ya > yb$.

(g) *If* $x \neq 0$ *then* $x^2 > 0$.

(h) *If* $0 < x < y$ *then* $0 < y^{-1} < x^{-1}$.

It is interesting that being an ordered field means that there is a copy of the rational numbers present. (Clearly this is a desired feature of the real numbers.)

Proposition 1.6 (Ordered fields contain $\mathbb{Q}$)**.** *Any ordered field* $\mathbb{F}$ *contains* $\mathbb{Q}$ *as a subfield, meaning that there is an injection* $\mathbb{Q} \to \mathbb{F}$ *such that the addition and multiplication operations induced on* $\mathbb{Q}$ *are the usual ones.*

Corollary 1.7. *An ordered field cannot be finite.*

Example 1.8.

(a) $\mathbb{Z}_p$ *is not an ordered field.*

(b) $\mathbb{Q}$ *is an ordered field.*

The following example is "more interesting."

Example 1.9 (Rational functions with rational coefficients)**.** *Let* $\mathbb{Q}[x]$ *be the collection of rational functions of the form*

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_0} \tag{1.1.1}$$

where n, m *are positive integers and* $a_n, \ldots, a_0, b_{m-1}, \ldots, b_0$ *are rational numbers. Under the usual notion of addition and multiplication,* $\mathbb{Q}[x]$ *forms an a field.*

We say that a function f *of the form* (1.1.1) *is* ***positive****, and write* $f > 0$*, if* $a_n > 0$*. Using this, we say that* $f > g$ *if* $f - g > 0$*. With this notion of order,* $\mathbb{Q}[x]$ *forms an ordered field.*

Exercise 1.1.

(a) Verify that $\mathbb{Q}[x]$ is an field. What is the additive identity? What is the multiplicative identity?

(b) Verify that $\mathbb{Q}[x]$ forms an ordered field.

Exercise 1.2.

(a) How should one define the symbols "$\leq$" and "$\geq$" in an ordered field?

(b) How should one define x^n, where $n \in \mathbb{N}$ and x is an element of an ordered field?

(c) Is it possible to define $x^{1/n}$, where $n \in \mathbb{N}$ and x is an element of an ordered field?

Exercise 1.3 (Bernoulli's inequality)**.** Suppose that $y > 0$ is an element of an ordered field. Use induction to prove that

$$(1+y)^n > 1 + ny.$$

holds for all $n \in \mathbb{N}$.

Exercise 1.4 (Binomial formula)**.** Use induction to show that the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

holds in an ordered field.

Exercise 1.5 (Geometric sum formula)**.** Use induction to show that the geometric sum formula

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

holds for any x in an ordered field.

Exercise 1.6 (Cauchy's inequality)**.** Prove that for all x, y in an ordered field that

$$x \cdot y \leq \frac{1}{2}\left(x^2 + y^2\right)$$

1.2 Absolute value

Frequently in calculus we make use of the absolute value, which can be defined using only the ordered field properties.

Definition 1.10 (Absolute value)**.** *The absolute value function $|\cdot|$ on an ordered field $\mathbb{F}$ is defined by*

$$|x| = \begin{cases} x & \text{if } x \geq 0. \\ -x & \text{if } x < 0. \end{cases}$$

Proposition 1.11 (Properties of absolute value)**.** *The absolute value function on an ordered field satisfies the following properties*

- *Positive definite:* $|x| \geq 0$*, and if* $|x| = 0$ *then* $x = 0$*.*
- *Symmetry:* $|-x| = |x|$*.*
- *Triangle inequality:* $|x + y| \leq |x| + |y|$*.*
- *Multiplicativity:* $|x \cdot y| = |x| \cdot |y|$*.*

Proposition 1.12 (Reverse triangle inequality)**.** *The absolute value function satisfies following estimate*

$$\big||x| - |y|\big| \leq |x - y|,$$

which is known as the "reverse triangle inequality."

Exercise 1.7. Show that for any x in an ordered field we have

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|x| \leq -x \leq |x|.$$

Exercise 1.8. Suppose that $|x| < \varepsilon$ for some $\varepsilon > 0$. Show that

$$-\varepsilon < x < \varepsilon.$$

Exercise 1.9. Suppose that $x \leq y \leq z$. Show that $|y| \leq |x| + |z|$.

Exercise 1.10. Prove the reverse triangle inequality holds in ordered fields.

1.3 Sequences

Now that we have defined the absolute value function, we can define what it means for a sequence to converge.

Definition 1.13.

(a) *A sequence* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{F}$ *is* ***bounded*** *if there exists* $C \in \mathbb{F}$ *with* $C > 0$ *such that* $|a_n| < C$ *for all* $n \in \mathbb{N}$*.*

(b) *A sequence* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{F}$ ***converges*** *to* $a \in \mathbb{F}$ *if for all* $\varepsilon \in \mathbb{F}$ *with* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$n > N \implies |a_n - a| < \varepsilon.$$

In this case we write $\lim_{n\to\infty} a_n = a$*, or simply* $a_n \to a$*.*

Proposition 1.14. *If $a_n \to a$ then $\{a_n\}$ is bounded.*

Proposition 1.15 (Properties of convergent sequences)**.** *Suppose $a_n \to a$ and $b_n \to b$ in $\mathbb{F}$. Then*

(a) $\lim_{n\to\infty} (a_n + b_n) = a + b$,

(b) $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$,

(c) *for all $x \in \mathbb{F}$ we have* $\lim_{n\to\infty} (x \cdot a_n) = x \cdot a$,

(d) *if $a_n \neq 0$ for all $n \in \mathbb{N}$ and $a \neq 0$, then*

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}.$$

Proposition 1.16 (Limit comparison for sequences)**.** *Suppose that $a_n \to a$ and $b_n \to b$, and that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then $a \leq b$.*

Proposition 1.17 (Squeeze Principle for Sequences)**.** *Suppose $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences such that*

$$a_n \leq b_n \leq c_n \quad \text{for all } n \in \mathbb{N} \tag{1.3.1}$$

and such that $a_n \to L$, $c_n \to L$ for some number L. Then $b_n \to L$.

Remark 1.18. *In Propositions 1.16 and 1.17, and in similar propositions, we don't actually need the estimates to hold for all $n \in \mathbb{N}$. Rather, they just need to hold for all $n \geq N$ for some fixed $N \in \mathbb{N}$.*

Exercise 1.11. Prove Propsition 1.14

Exercise 1.12. Prove Proposition 1.17.

Exercise 1.13. In this exercise you work within the field $\mathbb{Q}[x]$.

(a) Let $f(x) = \frac{1}{x}$ and $g(x) = 1$. Show that $0 < f < g$.

(b) Let be f, g as above and set $f_n = n \cdot f$. Show that the sequence $\{f_n\}$ is bounded by g. (That is, show that $f_n < g$ for all $n \in \mathbb{N}$.)

Exercise 1.14. In this exercise you work within the field $\mathbb{Q}[x]$. Let $f_n(x) = \frac{1}{n}$ and $g(x) = \frac{1}{x}$. Show that $f_n > g$ for all n. Explain how this shows that $\{f_n\}$ does not converge to zero.

Exercise 1.15. In this exercise you work within the field $\mathbb{Q}$. Let $f \in \mathbb{Q}$ such that $f > 0$ and set $f_n = n \cdot f$. Show that the sequences $\{f_n\}$ is not bounded. (That is, show for any $g \in \mathbb{Q}$ that there exists n such that $f_n > g$.)

Exercise 1.16. In this exercise you work within the field $\mathbb{Q}$. Let $f_n = \frac{1}{n}$. Show that $\{f_n\}$ converges to zero.

1.4 The Archimedean property

The last four exercises of the previous section show that if one is only working with the ordered field properties of the real numbers, then one cannot conclude that $\{1/n\}$ converges to zero (the way it does in the rational field). Those exercises also show that this issue is related to the issue of whether successive multiples of one positive number are eventually larger than any other fixed number. As a consequence of those exercises, we must assume that the real numbers have an additional property, called the "Archimedean property."

Definition 1.19 (Archimedean property)**.** *An ordered field $\mathbb{F}$ satisfies the* ***Archimedean property*** *if for each $x \in \mathbb{F}$ with $x > 0$ there exists $n \in \mathbb{N}$ with $n > x$. Such a field is called an* ***Archimedean ordered field****.*

Proposition 1.20 (Sequential characterization of Archimedean Property)**.** *An ordered field $\mathbb{F}$ is Archimedean if and only if the sequence $\{1/n\}_{n=1}^{\infty}$ converges to zero.*

Example 1.21.

(a) $\mathbb{Q}[x]$ *does not have the Archimedean property.*

(b) $\mathbb{Q}$ *does have the Archimedean property.*

One important consequence of the Archimedean property is that between every two real numbers there exists a rational number. This property is called the "density of $\mathbb{Q}$."

Proposition 1.22 (Density of $\mathbb{Q}$)**.** *Suppose x, y are distinct elements of an Archimedean field. Then there exists a rational number between x and y.*

Exercise 1.17. Prove directly that in the field $\mathbb{Q}[x]$ there cannot be a rational number q between the numbers $1/x$ and $2/x$.

1.5 Applications to sequences

The Archimedean property of the real numbers has several useful applications to sequences.

Proposition 1.23. *If $|x| < 1$ then the sequence $\{x^n\}$ converges to zero.*

Corollary 1.24 (Ratio test for sequences)**.** *Suppose that sequence* $\{a_n\}_{n=0}^{\infty}$ *is such that there exists a number* $r < 1$ *with*

$$\frac{|a_{n+1}|}{|a_n|} \leq r$$

for all $n \in \mathbb{N}$*. Then* $a_n \to 0$.

Exercise 1.18. Suppose $|x| < 1$ and define the sequence $\{S_n\}$ by

$$S_n = \sum_{k=0}^{n} x^k.$$

Show that

$$S_n \to \frac{1}{1-x}.$$

Exercise 1.19. Prove that the following sequences converge to zero:

(a) $a_n = \dfrac{1}{n+10}$

(b) $a_n = \dfrac{n}{2^n}$

(c) $a_n = \dfrac{2^n}{n!}$

(d) $a_n = \dfrac{n}{n^2+1}$

(e) $a_n = \displaystyle\sum_{k=n}^{2n} \frac{1}{2^n}$.

Exercise 1.20. Fix $p \in \mathbb{N}$ and define the sequence $\{a_n\}$ by

$$a_n = \frac{1}{n^p}.$$

Show that $a_n \to 0$.

Exercise 1.21.

(a) Give a definition of what it means for a sequence $\{a_n\}$ to *converge to infinity*, a notion we denote by either $a_n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.

(b) Use your definition to show that the sequence $a_n = n^2 + n$ converges to infinity.

1.6 Bisection search

Definition 1.25 (Intervals)**.** *Let $a, b \in \mathbb{F}$.*

(a) *An **open interval** is a set of one of the following forms:*

$$\begin{aligned} (a,b) &= \{x \in \mathbb{F} \mid a < x < b\}, && \textit{or} \\ (a,\infty) &= \{x \in \mathbb{F} \mid a < x\}, && \textit{or} \\ (\infty,b) &= \{x \in \mathbb{F} \mid x < b\}. \end{aligned}$$

By convention $(a,a) = \emptyset$.

(b) *An **closed interval** is a set of one of the following forms:*

$$\begin{aligned} [a,b] &= \{x \in \mathbb{F} \mid a \le x \le b\}, && \textit{or} \\ [a,\infty) &= \{x \in \mathbb{F} \mid a \le x\}, && \textit{or} \\ (\infty,b] &= \{x \in \mathbb{F} \mid x \le b\}. \end{aligned}$$

Note that $[a,a] = \{a\}$.

(c) *A half-open interval is a set of one of the following forms:*

$$\begin{aligned} [a,b) &= \{x \in \mathbb{F} \mid a \le x < b\}, && \textit{or} \\ (a,b] &= \{x \in \mathbb{F} \mid a < x \le b\}. \end{aligned}$$

(d) *The **width** of interval I is denoted by $|I|$ and defined by $|I| = |b - a|$ if I is one of (a,b), $(a,b]$, $[a,b)$, or $[a,b]$. Otherwise $|I|$ is undefined.*

The following definition describes properties that we desire of continuous functions.

Definition 1.26.

(a) *A function $f\colon [a,b] \to \mathbb{F}$ has the **intermediate value property in** $\mathbb{F}$ if for every y contained in the closed interval whose endpoints are $f(a)$ and $f(b)$ there exists $x_* \in [a,b]$ with $f(x_*) = y$.*

(b) *A function $f\colon [a,b] \to \mathbb{F}$ has the **extreme value property in** $\mathbb{F}$ if there exists $x_* \in [a,b]$ such that $f(x_*) \ge f(x)$ for all $x \in [a,b]$.*

Remark 1.27. *The intermediate value property is related to solving equations. For example, suppose we wanted to solve the equation*

$$x^2 = 2.$$

One approach is to define $f\colon [1,2] \to \mathbb{F}$ by $f(x) = x^2$. We have $f(1) = 1$ and $f(2) = 4$. The number 2 is contained in the closed interval whose endpoints are 1 and 4. Thus if we can show that f has the intermediate value property in $\mathbb{F}$, then we can deduce that there exists $x \in \mathbb{F}$ with $x^2 = 2$.

In both the case of the intermediate value property and the case of the extreme value property, one can go about looking for x_* using the method of *bisection search.*

Proposition 1.28 (Bisection search for intermediate values)**.** *Suppose*

$$f\colon [a,b] \to \mathbb{F}$$

and that y is contained in the closed interval whose endpoints are $f(a)$ and $f(b)$. Then there exists a sequence of closed intervals $I_n = [a_n, b_n] \subseteq [a,b]$ such that

- $I_n \supset I_{n+1}$ *for all* $n \in \mathbb{N}$,
- *for all $n \in \mathbb{N}$ we have y contained in the closed interval whose endpoints are $f(a_n)$ and $f(b_n)$, and*
- $|I_n| \to 0$.

Proposition 1.29 (Bisection search for extreme values)**.** *Suppose*

→ **Rephrase this just for sets – no need to have a function here**

$$f\colon [a,b] \to \mathbb{F}$$

is a bounded function. Then there exists a sequence of closed intervals $I_n = [m_n, M_n] \subset \mathbb{F}$ such that

- $I_n \supset I_{n+1}$ *for all* $n \in \mathbb{N}$,
- *for all $n \in \mathbb{N}$ we have both $f([a,b]) \cap I_n \neq \emptyset$ and $I_n \setminus f([a,b]) \neq \emptyset$, and*
- $|I_n| \to 0$.

Definition 1.30. *We say that* ***binary search is successful in*** $\mathbb{F}$ *if for every sequence of closed intervals I_n with*

- $I_n \supset I_{n+1}$ *for all* $n \in \mathbb{N}$ *and*
- $|I_n| \to 0$

we have $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$.

It is desirable to case the property that bisection search is successful in terms of sequences. We do this using the following definition.

Definition 1.31. *A sequence* $\{x_n\}$ *is* ***Cauchy*** *if for every* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$m, n \geq N \implies |x_n - x_m| < \varepsilon.$$

Proposition 1.32. *All convergent sequences are Cauchy.*

Proposition 1.33. *All Cauchy sequences are bounded.*

Proposition 1.34 (Sequential criterion for binary search)**.** *The following are equivalent:*

(a) *Binary search is successful in* $\mathbb{F}$*.*

(b) *All Cauchy sequences converge in* $\mathbb{F}$*.*

Exercise 1.22. Prove Proposition 1.29 (Binary search for extreme values).

Exercise 1.23. Prove that all Cauchy sequences are bounded.

Exercise 1.24. Show that if all Cauchy sequences converge in $\mathbb{F}$ then binary search is successful in $\mathbb{F}$.