Writing Assignment: SIR models

In this writing assignment, you consider the "SIR" model for infectious diseases. In this model, there is a population with N individuals. We let

- $S(t) =$ the number of individuals that is susceptible to the disease,
- $I(t) =$ the number of individuals infected by the disease, and
- $R(t) =$ the number of individuals removed from population by the disease.

At all times we have $S + I + R = N$. (Why?)

We assume the following differential equations govern how these quantities change in time:

$$\frac{dS}{dt} = -\beta IS, \qquad \frac{dI}{dt} = \beta IS - \gamma I, \qquad \frac{dR}{dt} = \gamma I.$$

Here β, γ are constants.

Task 1: Your first task is to describe in words the assumptions that these three differential equations represent.

Since we can solve for R by simply subtracting, it is enough to focus on the equations for S and I. Thus for the remainder of the activity, we consider the reduced system

$$\frac{dS}{dt} = -\beta IS, \qquad \frac{dI}{dt} = \beta IS - \gamma I.$$

Task 2: Your second task is to show that only equilibrium solutions to the reduced system is $(S, I) = (S_*, 0)$, where S_* is some number between 0 and N. (Why this restriction on?) Interpret these equilibrium solutions.

One important question in the study of the SIR model is the stability of the equilibrium at $(S, I) = (N, 0)$. It turns out that the stability of this equilibrium depends on the value of the constant

$$r_0 = \frac{\beta N}{\gamma},$$

which is called the *basic reproductive ratio* for the disease. In the book *Essential Mathematical Biology*, author N. Britton estimates that for smallpox we have $r_0 \approx 4$, while for measles we have $r_0 \approx 12$. (What are the units of r_0?)

Task 3: Determine for which values of r_0 the equilibrium $(N, 0)$ is stable and for which it is unstable.

- When $(N, 0)$ is unstable, we say that the population is "vulnerable." Have Sage generate a phase diagram for the linearized equation, and interpret your results about the linearized equation in terms of this model.

We now turn to analyzing the larger phase plane for this model. Explain why the only region of physical interest is when $S \geq 0$, $I \geq 0$, and $S + I \leq N$. What does this region look like in the phase space?

Task 4: Determine where are the nullclines of the system. Focus on the case where the population is vulnerable. The nullclines divide the region of physical interest in to two pieces. Determine whether I is increasing or decreasing in each region.

Suppose we are in a situation where a population is vulnerable, due to the value of r_0 for that disease. We are interested in preventing an epidemic by using vaccines. In practice, it is not possible to vaccinate 100% of the population. Let p be the percent of the population that is vaccinated. In order to learn whether the vaccination rate is high enough, we analyze the stability of the equilibrium at $(S, I) = ((1 - p_0)N, 0)$. Why is this the equilibrium we want to analyze?

Task 5: Working under the assumption that we are in a vulnerable state, analyze the stability of the equilibrium at $((1-p_0)N, 0)$. For which values of p will an epidemic emerge, and for which values will an epidemic be prevented? Describe the threshold in terms of the constant r_0.

Based on this model, what percent of the population must be vaccinated in order prevent a measles outbreak? What about smallpox?

Once you have completed all these tasks, write a paper that presents your analysis. You do not need to show all your work, but you need to provide enough detail that a classmate can follow your reasoning, and reproduce all the necessary details based on what you write. It is also helpful to explain how to interpret your various findings.