DAY 19

Linear theory: complex eigenvalues

ch:complex-eigenvalues

In this section we explore how to use complex eigenvalues and eigenvectors to construct real solutions to linear systems of equations. We begin with an example.

complex:long-example

EXAMPLE 19.1. *Consider the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix} Y.$$

In the previous section we found that there were two eigenvalues

$$\lambda_+ = 2 + 2i \qquad \text{and} \qquad \lambda_- = 2 - 2i$$

having the associated eigenvectors

$$\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}.$$

These give rise to complex solutions to the differential equation

$$Y_+(t) = e^{(2+2i)t}\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} \qquad \text{and} \qquad Y_-(t) = e^{(2-2i)t}\begin{pmatrix} 1 \\ 1-2i \end{pmatrix}.$$

A general complex solution to the equation is therefore

$$\begin{aligned} Y(t) &= Ae^{(2+2i)t}\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} + Be^{(2-2i)t}\begin{pmatrix} 1 \\ 1-2i \end{pmatrix} \\ &= Ae^{2t}\left\{\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + i\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}\right\} \\ &\quad + Be^{2t}\left\{\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} - i\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}\right\}. \end{aligned}$$

Notice that the real parts of $Y_+(t)$ and $Y_-(t)$ are the same, while the imaginary parts are opposites. We can use this to our advantage in order to construct real solutions.

First, choose $A = \frac{1}{2}$ and $B = \frac{1}{2}$. In this case we obtain the solution

$$Y_1(t) = e^{2t}\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix}.$$

Second, choose $A = \frac{1}{2i}$ *and* $B = -\frac{1}{2i}$*. In this case we obtain the solution*

$$Y_2(t) = e^{2t}\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}.$$

These two solutions are independent of one another, and are real-valued functions! Thus we have succeeded in constructing two real, independent solutions. Consequently, we can form a (real) general solution to the differential equation:

$$Y(t) = \alpha e^{2t}\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + \beta e^{2t}\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}.$$

We also want to understand what the phase portrait for this equation looks like. We do this in two steps. First we analyze the solution $Y_1(t)$*. Then we show that* $Y_2(t)$ *behaves in a qualitatively similar manner.*

To understand how $Y_1(t)$ *behaves we write it as*

$$Y_1(t) = e^{2t}\cos(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\sin(2t)\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Let's slowly build this function by modifying simpler functions.

(1) *The function*

$$\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

traverses the unit circle in the anti-clockwise direction; we can interpret this as alternating between having a position that is displaced from the origin in the

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

directions, with motion from the first vector to the second vector.

(2) *We can modify this unit circle trajectory by replacing the vectors*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with other vectors. Thus the trajectory

$$\cos(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

alternates between displacement in the

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

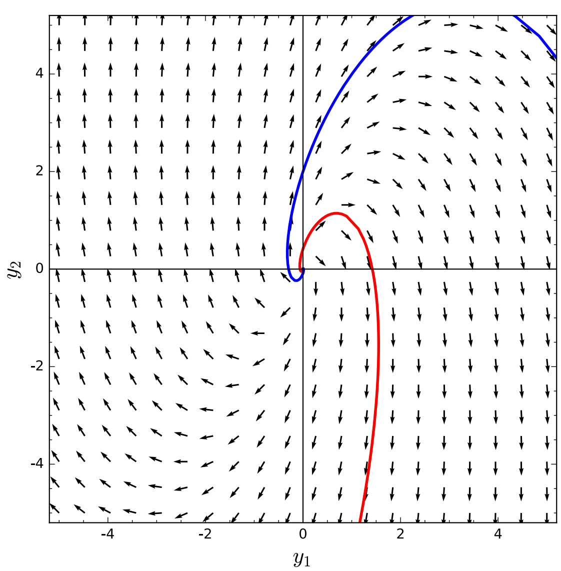
directions, with motion from the first to the second. The result is an elliptical path, traversed in the clockwise direction.

(3) *If we replace* $\cos(t)$ *and* $\sin(t)$ *by* $\cos(2t)$ *and* $\sin(2t)$*, the result is a trajectory that traverses the same path, but twice as fast.*

(4) *Finally, to obtain* $Y_1(t)$ *we multiply the whole thing by* e^{2t}*. This has the effect of increasing the overall location outwards as t increases, changing the elliptical trajectory to an outwards spiral trajectory.*

Now that we have analyzed $Y_1(t)$ *we examine* $Y_2(t)$*. While we could repeat the process used above to examine* $Y_1(t)$*, it is more efficient to simply note that* $Y_2(t) = e^{-\pi/2}Y_1(t - \pi/4)$*. Thus the two follow similar trajectories, but are out of phase of one another.*

The following shows the phase diagram for the differential equation, with $Y_1(t)$ *in red and* $Y_2(t)$ *in blue.*



We can learn several lessons from Example 19.1 that help us to more efficiently analyze differential equations in the case of complex eigenvalues.

The first is that complex eigenvalues, and the corresponding complex solutions, come in pairs where the real parts are the same and the imaginary parts are opposites. In particular, we always get two solutions such that

$$Y_+(t) = \operatorname{Re}(Y_+(t)) + i\operatorname{Im}(Y_+(t))$$
$$Y_-(t) = \operatorname{Re}(Y_+(t)) - i\operatorname{Im}(Y_+(t)).$$

Using the superposition principle, we know that

$$Y_1(t) = \mathrm{Re}(Y_+(t)) = \frac{1}{2}Y_+(t) + \frac{1}{2}Y_-(t)$$
$$Y_2(t) = \mathrm{Im}(Y_+(t)) = \frac{1}{2i}Y_+(t) + \frac{1}{2i}Y_-(t)$$

are solutions. Thus in order to find two real, independent solutions all we need to do is to find $Y_+(t)$ and then take, separately, the real and imaginary parts.

The second lesson that we learn from the example above is that if $\lambda = a + ib$ then the real solution $Y_1(t)$ takes the form

$$Y_1(t) = e^{at}\cos(bt)\begin{pmatrix}\heartsuit\\ \spadesuit\end{pmatrix} + e^{at}\sin(bt)\begin{pmatrix}\diamondsuit\\ \clubsuit\end{pmatrix}$$

for some constants $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$. Thus the solution $Y_1(t)$ will oscillate between being displaced in two different directions, rotating in either a clockwise or anticlockwise direction. Furthermore,

- If $\mathrm{Re}(\lambda) > 0$ then solutions spiral outwards. In this case, the equilibrium at $(0, 0)$ is called a ***spiral source***. Spiral sources are unstable.
- If $\mathrm{Re}(\lambda) = 0$ then solutions traverse an elliptical trajectory. In this case, the equilibrium $(0, 0)$ is called a ***center***. Centers are generally considered to be unstable, but interpretations vary.
- If $\mathrm{Re}(\lambda) < 0$ then solutions spiral inward towards $(0, 0)$. In this case, the equilibrium at $(0, 0)$ is called a ***spiral sink***. Spiral sinks are stable.

The third lesson we learn from Example 19.1 is that the two solutions Y_1 and Y_2 traverse trajectories that differ only by scaling and phase. Thus if we are only interested in obtaining a qualitative understanding, it is enough to study $Y_1(t)$.

Finally, note that there is an easy trick for seeing whether a spiral solution is rotating in the clockwise or anticlockwise direction. Consider the solution that passes through the point $(1, 0)$ at that point, the velocity vector is determined by the differential equation. If the vector is pointing in to the first quadrant, then the solution traverses anticlockwise; if the vector is pointing in to the fourth quadrant, then the solution is traversing in the clockwise direction.

EXAMPLE 19.2. *Consider the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix}1 & 2\\ -5 & -1\end{pmatrix}Y.$$

We compute the eigenvalues to be $\lambda = \pm 3i$. Already at this stage we know that solutions traverse elliptical paths around the center-type equilibrium at $(0,0)$. Furthermore, we know that the solution passing through the point $(1,0)$ has velocity vector

$$\begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

at that point. Thus solutions traverse in the clockwise direction.

In order to obtain a general solution, we compute the eigenvector corresponding to $\lambda = 3i$, which is

$$\begin{pmatrix} 2 \\ -1+3i \end{pmatrix}.$$

Thus the first complex solution is

$$Y_+(t) = e^{3i} \begin{pmatrix} 2 \\ -1+3i \end{pmatrix} = \begin{pmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} + i \begin{pmatrix} 2\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}$$

From this we may extract the real and imaginary parts, which we know are each real solutions:

$$Y_1(t) = \begin{pmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} \quad \textit{and} \quad Y_2(t) = \begin{pmatrix} 2\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}.$$

Consequently, the general solution is

$$Y(t) = \alpha \begin{pmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} + \beta \begin{pmatrix} 2\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}$$

Finally, examining $Y_1(t)$ we find

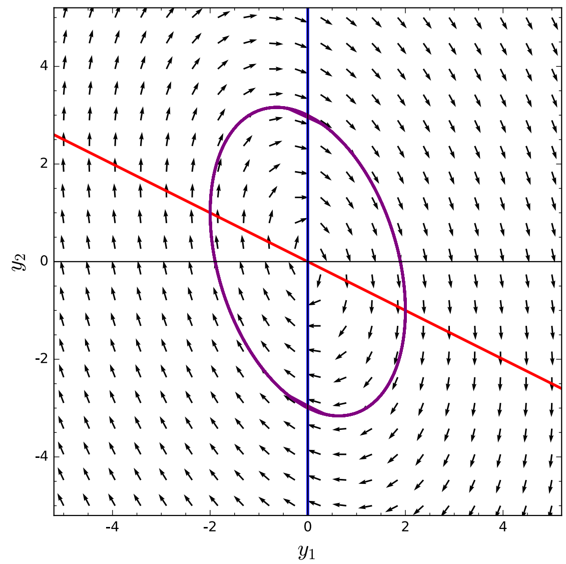
$$Y_1(t) = \cos(3t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \sin(3t) \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

Thus we see that solutions oscillate between displacements in the directions

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 \\ -3 \end{pmatrix}. \tag{19.1}$$

complex:example2-directio

The phase space diagram for this equation is as follows:



In the diagram, the trajectory of solution $Y_1(t)$ is shown in purple. The two directions (19.1) appear in red and blue, respectively.

ACTIVITY 19.1. *Analyze the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix} -8 & 13 \\ -2 & 2 \end{pmatrix} Y.$$

Determine the type of equilibrium point at $(0, 0)$ and the direction in which solutions traverse the phase plane. Use this to make a crude sketch of the phase diagram. Then find the general solution.

Exercises for Day 19

SolveComplexIVP

EXERCISE 19.1.

(1) *Find the explicit solution of the following IVP.*

$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

(2) *Suppose $x_0 = 1$ and $y_0 = 2$. What is the corresponding solution to the IVP?*

ComplexSystems

EXERCISE 19.2. *Study each of the following systems by addressing the following questions:*

- *Find the general solution of the system;*
- *Draw the phase diagram for the system without any use of "technology". Then check your answer with Sage.*
- *Draw the solution curves in the t–x and the t–y planes.*

- *Discuss the long-term fate of the solutions of the system. Your answer potentially depends on the initial condition $Y(0)$.*
- *Discuss the stability of the equilibrium solution $Y(t) = 0$.*

(1) $\dfrac{dY}{dt} = \begin{pmatrix} 6 & 9 \\ -5 & -6 \end{pmatrix} Y$;

(2) $\dfrac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix} Y$;

(3) $\dfrac{dY}{dt} = \begin{pmatrix} 5 & -3 \\ 6 & -1 \end{pmatrix} Y$.

ComplexEthel

EXERCISE 19.3. *A market researcher established that the daily profits of two competing stores, Ethel's Knick-knack Heaven and Irma's Antiques, relate to each other as in the system:*

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = -5x + 3y$$

here $x(t)$ denotes the daily profit of Ethel's store and $y(t)$ denotes the daily profit of Irma's store. The time is measured in months.

(1) *Draw the phase portrait of this system. Use the least amount of computations possible.*
(2) *Use the phase portrait only to discuss the long-term fate of these stores.*

LTExtra-First

EXERCISE 19.4. *Draw the phase diagrams of the following systems, using the least possible amount of computation.*

(1) $\dfrac{dY}{dt} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} Y$

(2) $\dfrac{dY}{dt} = \begin{pmatrix} -5 & 2 \\ -3 & 0 \end{pmatrix} Y$

(3) $\dfrac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} Y$

LTExtra-Last

EXERCISE 19.5. *The following problem concerns profits of two nearby stores; the model we shall use is based on a linear system of equations. The first store, the profit of which at time t we label by $x(t)$, was successful on its own until recently when a new store opened nearby. This second store, the profit of which at time t we label by $y(t)$, offers a lot of low-quality cheap merchandise. If it wasn't for the customers of the* *old store dropping by periodically the second store would not be able to survive.*

The model applicable to these two stores is:

$$\begin{cases} \dfrac{dx}{dt} = x - 5y \\ \\ \dfrac{dy}{dt} = 2x - y. \end{cases}$$

Currently, the "profits" of both stores are negative. Determine if the stores will ever recover, and what their long term fate is. Your supporting evidence should at least include a phase diagram.