DAY 18

Complex numbers

ch:complex-numbers

Suppose we want to find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix}. \tag{18.1}$$

complex:motivation

The characteristic equation for this matrix is

$$(1-\lambda)(3-\lambda)+5=0,$$

which we rewrite as

$$\lambda^2 - 4\lambda + 8 = 0.$$

The quadratic formula tells us that the solutions are

$$\lambda_+ = \frac{4+\sqrt{-16}}{2} \qquad \text{and} \qquad \lambda_- = \frac{4-\sqrt{-16}}{2}$$

Unfortunately, these are not real numbers. Thus the linear differential equation defined by the matrix (18.1) does not have straight line solution. However, this does not mean that all hope for using superposition to understand the differential equation is lost. Rather, we need to do a bit more work. That work involves using complex numbers, which is the subject of this section.

Complex numbers are specifically designed to ensure that all quadratic equations have solutions. This is accomplished by the inclusion of a new number i defined by $i^2 = -1$. A ***complex number*** is defined to be a number of the form

$$a + bi,$$

where both a and b are real numbers. The number a is called the ***real part*** of $a+bi$ and the number b is called the ***imaginary part*** of $a+bi$. Note that both the real part and the imaginary part are themselves real numbers. It is common to use the following notation for the real and imaginary parts of a complex number:

$$\mathrm{Re}(a+bi) = a \qquad \text{and} \qquad \mathrm{Im}(a+bi) = b.$$

EXAMPLE 18.1. *Consider the equation*

$$\lambda^2 + 3\lambda + 4 = 0.$$

From the quadratic formula we see that solutions are

$$\lambda_+ = \frac{-3+\sqrt{9-16}}{2} = -\frac{3}{2} + \frac{\sqrt{7}}{2} i$$
$$\lambda_- = \frac{-3-\sqrt{9-16}}{2} = -\frac{3}{2} - \frac{\sqrt{7}}{2} i.$$

Thus

$$\mathrm{Re}(\lambda_+) = -\frac{3}{2} \qquad \textit{and} \qquad \mathrm{Im}(\lambda_+) = \frac{\sqrt{7}}{2}.$$

Complex numbers are a very interesting and important part of mathematics. I strongly encourage all of you to take our Complex Variables course to learn more. For now, however, I will present only those few results that we need for this course.

First, note that we can perform all of the "usual" algebraic procedures on complex numbers. Adding/subtracting and multiplying/dividing is rather straightforward, provided we remember that $i^2 = -1$. For example,

$$(2-3i) + (7+2i) = 9 - i,$$
$$(2-3i)(7+2i) = 14 - 17i - 6i^2 = 20 - 17i.$$

One can also define what it means to compute the exponential function of a complex number by means of the Taylor series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots.$$

In particular, we have

$$\begin{aligned} e^{a+ib} &= e^a e^{ib} \\ &= e^a \left[1 + (ib) + \frac{1}{2}(ib)^2 + \frac{1}{6}(ib)^3 + \frac{1}{24}(ib)^4 + \dots\right] \\ &= e^a \left[\left(1 - \frac{1}{2}b^2 + \frac{1}{24}b^4 - \dots\right) + i\left(b - \frac{1}{6}b^3 + \dots\right)\right] \\ &= e^a \left(\cos(b) + i\sin(b)\right). \end{aligned}$$

In the case that $a = 0$ this identity becomes

$$e^{ib} = \cos(b) + i\sin(b)$$

and is known as ***Euler's formula***.

Euler's formula has several interesting and useful applications.

ACTIVITY 18.1. *Use Euler's formula to deduce that*

$$e^{-ib} = \cos(b) - i\sin(b).$$

Conclude from this that

$$\cos(b) = \frac{e^{ib} + e^{-ib}}{2} \qquad \textit{and} \qquad \sin(b) = \frac{e^{ib} - e^{-ib}}{2}.$$

ACTIVITY 18.2. *Since* $e^{2\theta} = (e^{\theta})^2$, *Euler's formula implies that*

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2.$$

Multiply out the right side of this identity in order to deduce the double angle formulas.

ACTIVITY 18.3. *Use the fact that* $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$, *together with Euler's formula, to deduce the angle sum formulas.*

Finally, we return to the problem that motivated this discussion: the eigenvalues of the matrix (18.1). The computation above shows that the eigenvalues are

$$\lambda_+ = 2 + 2i \qquad \text{and} \qquad \lambda_- = 2 - 2i.$$

We can subsequently compute the eigenvectors to be

$$\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}.$$

In the next section we discuss how to use complex eigenvalues and eigenvectors in order to obtain real solutions to linear systems of equations.

Exercises for Day 18

EXERCISE 18.1. *Put the following expressions into the form* $a + b\,i$, *where a,b are real numbers.*

(1) $(1+2i)(3-4i)$

(2) $\dfrac{1}{2+i} + \dfrac{1}{1-2i}$

(3) $\dfrac{2+3i}{1+i}$

EXERCISE 18.2. *Solve the quadratic equation* $x^2 + x + 1 = 0$ *within the set of complex numbers.*

EXERCISE 18.3. *Solve the system of equations:*

$$x + iy = 2$$
$$2ix - y = 3i.$$

EXERCISE 18.4. *Find the eigenvalues and the eigenvectors of the matrix*

$$\begin{pmatrix} -2 & -9 \\ 1 & -2 \end{pmatrix}.$$

EXERCISE 18.5. *Derive the Euler formula using Taylor expansions.*

ComplexDecompose

EXERCISE 18.6. *Decompose into real and imaginary parts.*

(1) $e^{\frac{i\pi}{4}}$
(2) $e^{-1+\pi i}$
(3) $2e^{1+i} + 2e^{1-i}$
(4) $e^{-(2+\pi i)t}$
(5) $e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} - e^{-it}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$
(6) $e^{(1+i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} + e^{(1-i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

EXERCISE 18.7. *Assume that a, b are some real numbers with $b \neq 0$.*

(1) *Identify the real and the imaginary part of the expression $e^{(a+ib)t}$.*
(2) *Treat the real and the imaginary part as individual functions of the independent t variable: $f(t)$ and $g(t)$. Graph these two functions. Note that the graphs will look radically different depending on whether a is positive, zero, or negative. Explore all three situations.*