DAY 17

Linear theory: distinct real eigenvalues

ch:real-eigenvalues

In this section we study linear equations

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

in the case that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has two distinct, real eigenvalues $\lambda_1 < \lambda_2$. There are three general situations, and one special situation, that we consider:

- the positive case, when $0 < \lambda_1 < \lambda_2$;
- the negative case, when $\lambda_1 < \lambda_2 < 0$;
- the mixed case, when $\lambda_1 < 0 < \lambda_2$; and
- the zero case, when either $\lambda_1 = 0$ or $\lambda_2 = 0$.

Positive case. In the case that both eigenvalues are positive, both straight line solutions

$$Y_1(t) = e^{\lambda_1 t}\begin{pmatrix} \heartsuit \\ \spadesuit \end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{\lambda_2 t}\begin{pmatrix} \diamondsuit \\ \clubsuit \end{pmatrix}$$

are moving away from the equilibrium at $(y_1, y_2) = (0, 0)$ as t increases. Consequently, the general solution

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t).$$

moves away from the equilibrium as well. In this situation we say that the equilibrium $(0, 0)$ is a ***source***; source equilibria are unstable.

We are assuming that $0 < \lambda_1 < \lambda_2$. This means when $t \gg 0$ we have $e^{\lambda_1 t} \ll e^{\lambda_2 t}$. Thus as $t \to \infty$ the solution $Y_2(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_2(t)$.

Similarly, when $t \ll 0$ we have $e^{\lambda_1 t} \gg e^{\lambda_2 t}$. Thus as $t \to -\infty$, the solution $Y_1(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_1(t)$.

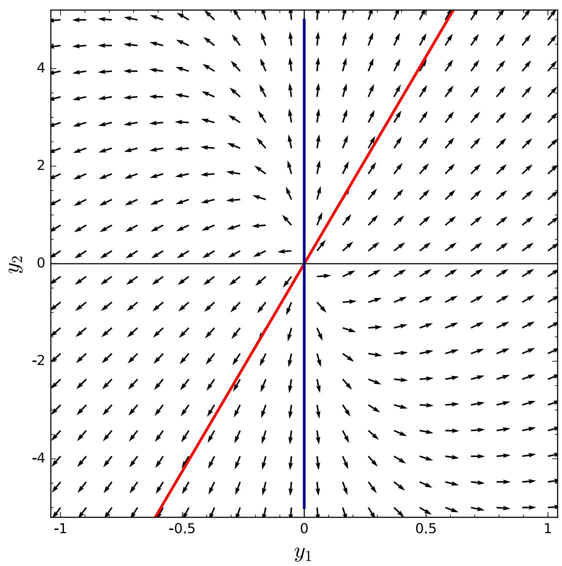
EXAMPLE 17.1. *Consider the equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 5 & 0 \\ 17 & 3 \end{pmatrix} Y.$$

We compute the eigenvalues to be $\lambda_1 = 3$ *and* $\lambda_2 = 5$. *The corresponding straight line solutions are*

$$Y_1(t) = e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \textit{and} \qquad Y_2(t) = e^{5t}\begin{pmatrix} 2 \\ 17 \end{pmatrix}.$$

The phase diagram for this system is:



Here $Y_1(t)$ *(and its negative) appears in blue, while* $Y_2(t)$ *(and its negative) appears in red. Notice that as* $t \to \infty$, *typical solutions move parallel to* $Y_2(t)$, *while as* $t \to -\infty$ *solutions move parallel to* Y_1.

Negative case. In the case that both eigenvalues are negative, both straight line solutions

$$Y_1(t) = e^{\lambda_1 t}\begin{pmatrix} \heartsuit \\ \spadesuit \end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{\lambda_2 t}\begin{pmatrix} \diamondsuit \\ \clubsuit \end{pmatrix}$$

are moving towards the equilibrium at $(y_1, y_2) = (0, 0)$ as t increases. Consequently, the general solution

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t).$$

moves towards the equilibrium as well. In this situation we say that the equilibrium $(0, 0)$ is a ***sink***; source equilibria are stable.

We are assuming that $\lambda_1 < \lambda_2 < 0$. This means when $t \gg 0$ we have $e^{\lambda_1 t} \ll e^{\lambda_2 t}$. Thus as $t \to \infty$ the solution $Y_2(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_2(t)$.

Similarly, when $t \ll 0$ we have $e^{\lambda_1 t} \gg e^{\lambda_2 t}$. Thus as $t \to -\infty$, the solution $Y_1(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_1(t)$.

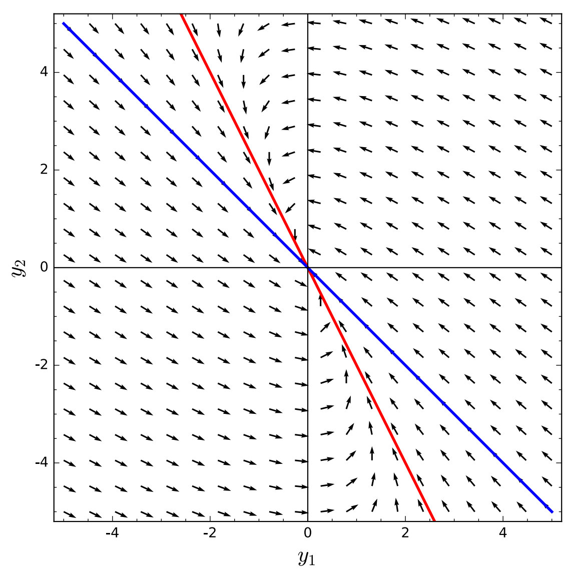
EXAMPLE 17.2. *Consider the example*

$$\frac{d}{dt}Y = \begin{pmatrix} -6 & -2 \\ 4 & 0 \end{pmatrix} Y.$$

We compute the straight line solutions to be

$$Y_1(t) = e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \textit{and} \qquad Y_2(t) = e^{-2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The phase diagram for this system is:



Here $Y_1(t)$ (and its negative) appears in blue, while $Y_2(t)$ (and its negative) appears in red. Notice that as $t \to \infty$, typical solutions move parallel to $Y_2(t)$, while as $t \to -\infty$ solutions move parallel to Y_1.

Mixed case. We now consider the case where $\lambda_1 < 0 < \lambda_2$. In this situation, the straight line solution

$$Y_1(t) = e^{\lambda_1 t}\begin{pmatrix} \heartsuit \\ \spadesuit \end{pmatrix}$$

is moving towards the equilibrium at $(y_1, y_2) = (0, 0)$ as t increases, while the straight line solution

$$Y_2(t) = e^{\lambda_2 t}\begin{pmatrix}\diamondsuit \\ \clubsuit\end{pmatrix}$$

is moving away from the equilibrium. Consequently, the general solution

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t)$$

ultimately moves away from the equilibrium point. In this situation we way that the equilibrium point $(0, 0)$ is a ***saddle***; saddle equilibria are unstable.

Notice that when $t \gg 0$ we have $e^{\lambda_1 t} \approx 0$. Thus as $t \to \infty$, the solution $Y_1(t)$ approaches zero and any general solution $Y(t)$ approaches $Y_2(t)$.

Similarly, when $t \ll 0$ we have $e^{\lambda_2 t} \approx 0$. Thus as $t \to -\infty$, the solution $Y_2(t)$ approaches zero and any general solution $Y(t)$ approaches $Y_1(t)$.

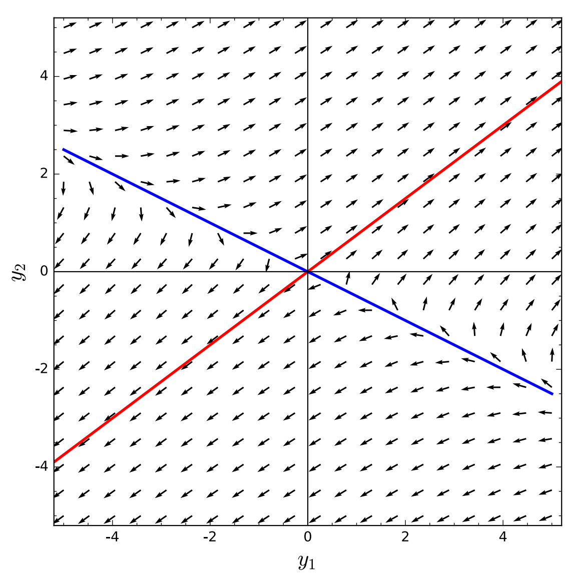
EXAMPLE 17.3. *Consider the equation*

$$\frac{d}{dt}Y = \begin{pmatrix}3 & 8 \\ 3 & 5\end{pmatrix} Y.$$

We compute the straight line solutions to be

$$Y_1(t) = e^{-t}\begin{pmatrix}-2 \\ 1\end{pmatrix} \qquad \textit{and} \qquad Y_2(t) = e^{9t}\begin{pmatrix}4 \\ 3\end{pmatrix}.$$

The phase diagram for the system is:



Notice that as $t \to \infty$ all solutions approach either $Y_2(t)$ or its negative, which appear in red. As $t \to -\infty$, solutions all approach either $Y_1(t)$ or its negative, which appear in blue.

Zero case. There are actually two different possible ways one eigenvalue could be zero:

$$\lambda_1 < \lambda_2 = 0 \qquad \text{or} \qquad 0 = \lambda_1 < \lambda_2.$$

We treat the second case, and leave the first case as an exercise.

Suppose, then that $\lambda_1 = 0$ and $\lambda_2 > 0$. This means that the second straight line solution takes the form

$$Y_2(t) = e^{\lambda_2 t} \begin{pmatrix} \heartsuit \\ \spadesuit \end{pmatrix},$$

while the second straight line solution takes the form

$$Y_1(t) = \begin{pmatrix} \diamondsuit \\ \clubsuit \end{pmatrix}.$$

Notice that $Y_1(t)$ is actually an equilibrium solution! Since any multiple of $Y_1(t)$ is also a solution, we see that there are a whole line of equilibrium solutions in the phase plane.

Since $\lambda_2 > 0$, we have $e^{\lambda_2 t} \to \infty$ as $t \to \infty$. This means that as $t \to \infty$, general solutions move away from the line of equilibrium solutions as $t \to \infty$.

The line of equilibrium solutions passes through the equilibrium at $(0, 0)$. Thus there are solutions nearby to $(0, 0)$ that neither move away, nor move closer. However, "almost all" nearby solutions move away from the equilibrium – only those on the line of equilibrium points don't. The likelihood of a perturbation away from $(0, 0)$ to land on the line of equilibrium points is essentially zero. Thus we say that $(0, 0)$ is unstable.

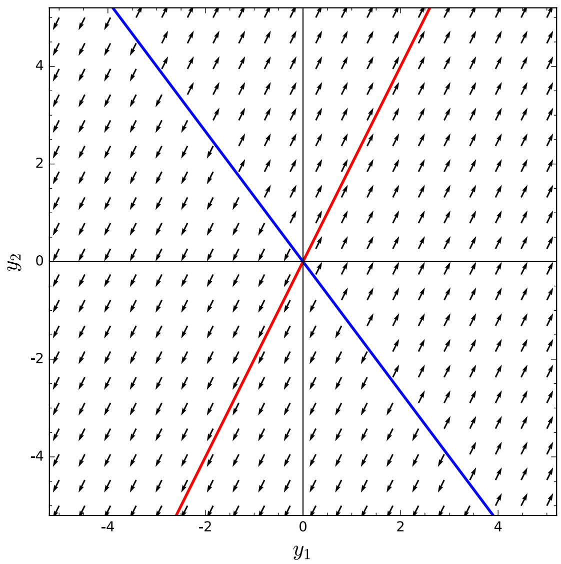
Example 17.4. *Consider the equation*

$$\frac{d}{dt} Y = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} Y.$$

We compute that the straight line solutions are

$$Y_1(t) = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{10t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The phase diagram for this equation is the following:



Notice the line of equilibrium solutions (in blue). Furthermore, all solutions not on that line are moving parallel to the non-constant solution $Y_2(t)$.

Exercises for Day 17

Real-First

EXERCISE 17.1. *Study each of the following "systems" by addressing the following questions:*

- *Find the general solution of the system;*
- *Draw the phase diagram for the system without any use of "technology". Then check your answer with Sage.*
- *Discuss the long-term fate of the solutions of the system. Your answer potentially depends on the initial condition* $Y(0)$.
- *Discuss the stability of the equilibrium solution* $Y(t) = 0$.

(1) $\dfrac{dY}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} Y$

(2) $\dfrac{dY}{dt} = \begin{pmatrix} 6 & 9 \\ 3 & 0 \end{pmatrix} Y;$

(3) $\dfrac{dY}{dt} = \begin{pmatrix} 0.41 & 0.12 \\ 0.12 & 0.34 \end{pmatrix} Y;$

(4) $\dfrac{dY}{dt} = \begin{pmatrix} 4 & 15 \\ -2 & -7 \end{pmatrix} Y;$

Ethel-and-Irma

EXERCISE 17.2. *A market researcher established that the daily profits of two competing stores, Ethel's Knick-knack Heaven and Irma's*

Antiques, relate to each other as in the system:

$$\frac{dx}{dt} = 6x - 2y$$
$$\frac{dy}{dt} = -2x + 3y$$

here $x(t)$ denotes the daily profit of Ethel's store and $y(t)$ denotes the daily profit of Irma's store. The time is measured in months.

(1) *Draw the phase portrait of this system. Use the least amount of computations possible.*
(2) *Use the phase portrait only to discuss the long-term fate of these stores if their current profits are:*
 (a) $x_0 = \$90$ *and* $y_0 = \$240$
 (b) $x_0 = \$100$ *and* $y_0 = \$120$
 (c) $x_0 = \$95$ *and* $y_0 = \$180$
(3) *Use the phase portrait to find the relationship between the current profits which would allow both stores to stay in business.*
(4) *Find (analytically) the particular solution of the model corresponding to the current profits of* $x_0 = \$90$ *and* $y_0 = \$240$*. Use the knowledge of this particular solution to determine when Ethel's profits are going to be biggest. How big is this profit?*
(5) *Do the same for Irma's store and the current profits of* $x_0 = \$95$ *and* $y_0 = \$180$.

Real-Last

EXERCISE 17.3. *Two* 100 *gallon mixing tanks $TANK_1$ and $TANK_2$ are connected to each other with two pipes, $PIPE_1$ and $PIPE_2$. The tanks are both completely filled with salty water. The salty water from $TANK_1$ flows through $PIPE_1$ to $TANK_2$ at the (continuous) rate of* $8\ ^{\text{gal}}/_{\text{hr}}$*. The salty water from $TANK_2$ flows through $PIPE_2$ to $TANK_1$ at the (continuous) rate of* $2\ ^{\text{gal}}/_{\text{hr}}$*. The volume of $TANK_1$ is kept constant by continuous adding of pure water at the rate of* 6 *gallons per hour. Likewise, the volume of $TANK_2$ is kept constant by continuous draining at the rate of* 6 *gallons per hour. Everything is always kept 'perfectly well mixed.'*

(1) *Let $s_1(t)$ and $s_2(t)$ be the amount of salt (in pounds, say) in $TANK_1$ and $TANK_2$ (respectively). Write a linear system of differential equations modeling $s_1(t)$ and $s_2(t)$.*
(2) *Draw the phase portrait of this system.*
(3) *What can you say about "the fate of" s_1 and s_2? Is it true that*

$$\lim_{t\to+\infty} s_1(t) = \lim_{t\to+\infty} s_2(t) = 0 \quad ?$$

Which of the two tanks will have more salt in the long run? Does your answer depend on how much salt the tanks initially had?

Sage code

Here is the Sage code used for making the plots above.

```
var('x,y,t')
Y = vector([x,y])
A = matrix([[2,1],[1,3]])

Diag,EV = A.eigenmatrix_right()
par1 = parametric_plot(t*EV.transpose()[0],(t,-10,10),
   thickness =2, color='red')
par2 = parametric_plot(t*EV.transpose()[1],(t,-5,5),
   thickness =2, color='blue')

Field = A*Y/(A*Y).norm()
FieldPlot = plot_vector_field(Field,(x,-5,5),(y,-5,5))

MainPlot = FieldPlot + par1 + par2
MainPlot.show(axes_labels=["$y_1$","$y_2$"],xmax=5,
   ymax=5,xmin=-5,ymin=-5)
```