DAY 16

Eigenstuff and straight-line solutions

ch:eigenstuff

In the previous section we discovered a recipe for constructing solutions to linear equations. In this section we address the first step of that procedure, finding two independent solutions Y_1 and Y_2 to our linear system.

To motivate our approach, let's recall the example appearing in Activity 15.4. In that case, we were presented with two solutions:

$$Y_1(t) = e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{2t} \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Notice that both of these solutions trace out straight lines in the phase plane; a plot of them is here:

[graphic needed]

This example motivates an approach to finding solutions: We look for ***straight line solutions*** of the form

$$Y(t) = s(t) \begin{pmatrix} x \\ y \end{pmatrix}, \tag{16.1}$$

straight-line-ansatz

where $s(t)$ is some function and x, y are fixed constants. Clearly not all solutions to linear equations are straight line solutions. But at this stage we don't need to construct all solutions – all we need to do is find some pair of independent solutions. So it makes sense to go looking for solutions that take a simple form, such as traversing a straight line.

Suppose, therefore, that we are looking for solutions to the linear equation

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{16.2}$$

eigen:generic-ode

If we plug in a generic straight line function of the form (16.1) we find that we need

$$s'(t)\begin{pmatrix} x \\ y \end{pmatrix} = s(t)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We can assume that $s(t)$ is not the zero function (because otherwise we don't get an interesting solution) and rewrite this equation as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{s'(t)}{s(t)}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{16.3}$$

straight-ansatz-result

We now make an interesting observation: The left side of (16.3) does not depend on t: the entries of the matrix are constant and the entries of the vector are constant. Since the left side is constant in time, the right side must be as well. Thus we conclude the following: If(16.1) is going to be a straight line solution to (16.2), then it must be the case that $s'(t)/s(t)$ is constant. It is traditional to label this constant with the Greek letter λ. Replacing $s'(t)/s(t)$ by λ, the condition (16.3) becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}, \tag{16.4}$$

eigenvalue-problem

together with the condition that

$$\frac{s'(t)}{s(t)} = \lambda. \tag{16.5}$$

scaling-ode

Notice something amazing: the equation (16.4) does not involve any calculus…or even any functions! It is simply an algebraic equation where there are three unknowns: x, y, and λ. In fact, (16.4) is really just two equations with three unknowns. So it seems reasonable that we would be able to find several solutions. In other words, the prospects of finding straight-line solutions to (16.2) seem rather good.

The discussion in the previous paragraph suggests an approach for finding straight line solutions: First, find

$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \lambda$$

that satisfy (16.4). Second, using that value of λ, find a function $s(t)$ that satisfies (16.5). Finally, building our straight line solution using the formula (16.1) that we started with.

Before attempting to execute this procedure to specific examples, it is worth taking a moment to recall why we are seeking straight line solutions in the first place. Remember that our goal is to construct a general solution to (16.2) of the form

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t), \tag{16.6}$$

eigen:linear-combo

where $Y_1(t)$ and $Y_2(t)$ are independent solutions. Our plan is to find $Y_1(t)$ and $Y_2(t)$ that are straight line solutions of the form (16.1) by solving (16.4) and (16.5). Since in (16.6) we will be multiplying Y_1 and Y_2 by arbitrary constants α and β, we don't need to worry about including any free constants in our scaling solution. This implies that we can take the solution to (16.5) to be simply

$$s(t) = e^{\lambda t}$$

and that all the difficult work involves finding solutions to (16.4). Furthermore, since we want $Y_1(t)$ and $Y_2(t)$ to be independent, we do not want both x and y in (16.4) to be zero.

We now turn to the problem of finding x, y, λ satisfying (16.4), where we require that we don't have both x and y zero. This problem is called the ***eigenvalue problem*** for the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{16.7}$$

eigen:basic-matrix

We write equation (16.4) as

$$\begin{aligned} (a-\lambda)x + by &= 0 \\ cx + (d-\lambda)y &= 0 \end{aligned} \quad \leftrightarrow \quad \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16.8}$$

simplified-eigenvalue-pro

We now take a moment to study equations of this form; for convenience, we set $A = a - \lambda$, $B = b$, $C = c$, $D = (d - \lambda)$ so that the system is

$$\begin{aligned} Ax + By &= 0 \\ Cx + Dy &= 0. \end{aligned} \quad \leftrightarrow \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16.9}$$

generic-linear-system

By multiplying the first equation by D and the second equation by B, and then subtracting, we find that

$$(AD - BC)x = 0.$$

Similarly by multiplying the first by C and the second by A, and then subtracting, we find that

$$(AD - BC)y = 0.$$

Thus we see that in order to have x and y satisfying (16.9) and have at least one of x or y be not zero, then we must have

$$AD - BC = 0.$$

The quantity $AD - BC$ is called the ***determinant*** of the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

What we have just discovered is that in order to have a non-zero solution to (16.9) the determinant of the matrix must be zero.

We now apply this knowledge to (16.8). In order to have one of x or y not equal to zero, we must have

$$(a - \lambda)(d - \lambda) - bc = 0. \tag{16.10}$$

characteristic-equation

The equation (16.10) is called the ***characteristic equation*** for the matrix (16.7).

The solutions λ to the characteristic equation are called the ***eigenvalues*** of the matrix. The eigenvalues are precisely the values of λ for which we can find solutions x, y to (16.4) where at least one of x, y is not zero. Thus in order to construct our straight line solutions, we proceed as follows:

(1) Find the eigenvalues λ of the matrix appearing in our linear equation.
(2) For each eigenvalue λ, find a corresponding non-zero vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

satisfying (16.4). This vector is called an ***eigenvector*** associated to eigenvalue λ.
(3) For each eigenvalue, let $s(t) = e^{\lambda t}$. Use this function, together with the associated eigenvector, to construct the straight line solution as in (16.1).

Assuming that this procedure yields two independent straight line solutions, then we are able to construct a general solution to the differential equation using the superposition principle.

Example 16.1. *Consider the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y.$$

The characteristic equation for the matrix is

$$(2-\lambda)^2 - 1 = 0.$$

The solutions to the characteristic equation are

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 1.$$

We focus on $\lambda_1 = 3$. *In this case* (16.4) *becomes*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \leftrightarrow \qquad \begin{matrix} 2x + y = 3x \\ x + 2y = 3y \end{matrix}$$

Both of these equations reduce to $x = y$. *Thus we choose our eigenvector to be*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $s_1(t) = e^{\lambda_1 t} = e^{3t}$, the corresponding straight line solution is

$$Y_1(t) = e^{3t}\begin{pmatrix}1\\1\end{pmatrix}.$$

We now focus on $\lambda_1 = 1$. In this case (16.4) becomes

$$\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 1\begin{pmatrix}x\\y\end{pmatrix} \qquad \leftrightarrow \qquad \begin{aligned}2x + y &= x\\ x + 2y &= y\end{aligned}$$

Both of these equations reduce to $x = -y$. Thus we choose our eigenvector to be

$$\begin{pmatrix}-1\\1\end{pmatrix}.$$

Since $s_2(t) = e^{\lambda_2 t} = e^{t}$, the corresponding straight line solution is

$$Y_2(t) = e^{t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Since the two functions s_1 and s_2 do not simultaneously vanish, and since the two eigenvectors are independent, we have have constructed two independent straight line solutions. Using the superposition principle, we conclude that a general solution to the differential equation is

$$Y(t) = \alpha e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + \beta e^{t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

ACTIVITY 16.1. *Find the general solution to the system*

$$\frac{d}{dt}Y = \begin{pmatrix}4 & 7\\5 & 6\end{pmatrix}Y$$

ACTIVITY 16.2. *Find the general solution to the system*

$$\frac{d}{dt}Y = \begin{pmatrix}9 & 4\\5 & 1\end{pmatrix}Y$$

We have now developed an approach for finding solutions to linear systems of equations. Our plan now is to use this approach in order to understand the phase space diagrams for these systems. In particular, we want to use the method of “eigenstuff” to understand the stability of the equilibrium point at $(y_1, y_2) = (0, 0)$. As you will soon see, the stability of this equilibrium is determined primarily by the eigenvalues of the matrix that determines the equation. These eigenvalues, of course, are the solutions to the characteristic equation (16.10). Since the characteristic equation is quadratic, we see that there are a number of cases to consider: the case of two real solutions, the case of one real solution, and the case of no real solutions. In the next several sections we systematically deal with each of these cases, beginning with the case when there are two distinct real eigenvalues.

Exercises

FindThoseEigenvalues

EXERCISE 16.1. *Find the eigenvalues and the corresponding eigenvectors of the following matrices:*

(1) $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ *(2)* $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ *(3)* $\begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix}$

EStuff-Last

EXERCISE 16.2. *Find the explicit solution of the following IVP.*

$$\frac{dY}{dt} = \begin{pmatrix} 11 & 30 \\ -4 & -11 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} 8 \\ -3 \end{pmatrix}.$$