DAY 15

Tools for linear systems

We say that a differential equation is a ***linear system*** if it takes the form

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1 \begin{pmatrix} A \\ B \end{pmatrix} + y_2 \begin{pmatrix} C \\ D \end{pmatrix}, \tag{15.1}$$

where A, B, C, D are real numbers. We can write linear systems in the form

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} Ay_1 + Cy_2 \\ By_1 + Dy_2 \end{pmatrix}. \tag{15.2}$$

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In order to more systematically organize our theory for such equations, we introduce "matrix notation."

Matrix notation

Consider the right hand side of (15.2),

$$y_1 \begin{pmatrix} A \\ B \end{pmatrix} + y_2 \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} Ay_1 + Cy_2 \\ By_1 + Dy_2 \end{pmatrix}. \tag{15.3}$$

The essential content is given by the four numbers A, B, C, D. The idea of matrix notation is to put these four numbers in to a grid, called a ***matrix***, and view this right hand side as that matrix multiplying the vector containing y_1, y_2. To do this we the matrix

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

and the column vector

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then we define the product MY by

$$MY = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} Ay_1 + Cy_2 \\ By_1 + Dy_2 \end{pmatrix}. \tag{15.4}$$

There is a fun way to remember this formula by waving your hands in the air. . . ask Paul for details!

EXAMPLE 15.1.

$$\begin{pmatrix} 5 & -2 \\ 7 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5y_1 - 2y_2 \\ 7y_1 \end{pmatrix}.$$

ACTIVITY 15.1. *Multiply the following matrices and vectors*

(1)

$$\begin{pmatrix} 8 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

It is also important to be able to reverse engineer the matrix-vector decomposition.

ACTIVITY 15.2. *Express each of the following as a matrix times a vector.*

(1)

$$\begin{pmatrix} u + 3v \\ 5u + 2v \end{pmatrix}$$

(2)

$$\begin{pmatrix} p + 7q \\ 2p - q \end{pmatrix}$$

Using matrix and notation we can write equations of the form (15.2) as

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{15.5}$$

generic-linear-system

or simply as

$$\frac{d}{dt}Y = MY. \tag{15.6}$$

Activity 15.3. *Write the system*

$$\frac{dy_1}{dt} = y_2 \qquad \frac{dy_2}{dt} = y_1 + y_2$$

in vector-matrix notation.

The superposition principle

There are two important properties of matrix multiplication:

Matrix multiplication respects addition of vectors: This means that for any two vectors X and Y and any matrix M we have

$$M(X + Y) = MX + MY.$$

Matrix multiplication respects scaling of vectors: This means that for any vector X, any number α, and any matrix M we have

$$M(\alpha X) = \alpha MX.$$

Notice that the derivative operator has analogous properties:

Differentiation respects addition of vectors: This means that for any two vectors X and Y we have

$$\frac{d}{dt}(X + Y) = \frac{d}{dt}X + \frac{d}{dt}Y.$$

Differentation respects scaling of vectors: This means that for any vector X and any number α we have

$$\frac{d}{dt}(\alpha X) = \alpha\frac{d}{dt}X.$$

The fact that both matrix multiplication and differentiation respect addition and scaling gives us a way to generate new solutions to linear differential equations from old solutions. This method is called the ***superposition principle***.

Superposition principle: Suppose $Y_1(t)$ and $Y_2(t)$ are solutions to the linear equation

$$\frac{d}{dt}Y = MY.$$

Then for any numbers α and β we have that

$$\alpha Y_1(t) + \beta Y_2(t)$$

is also a solution.

The quantity $\alpha Y_1 + \beta Y_2$ is called a ***linear combination*** of Y_1 and Y_2.

The superposition principle is a very powerful tool.

perposition-mega-activity

ACTIVITY 15.4. *Consider the equation*

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(1) *Show that* $Y_1(t) = e^{5t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *is a solution.*
(2) *Show that* $Y_2(t) = e^{2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ *is a solution.*
(3) *Show that* $Y(t) = 3Y_1(t) + 7Y_2(t)$ *is a solution.*
(4) *Explain why* $Y(t) = \alpha Y_1(t) + \beta Y_2(t)$ *is a solution for any* α *and* β*.*
(5) *Show that* $Y_1(0)$ *and* $Y_2(0)$ *are independent. Find* α *and* β *such that*

$$\alpha Y_1(0) + \beta Y_2(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

(6) *With* α *and* β *as in the previous part, what initial value problem does* $Y(t) = \alpha Y_1(t) + \beta Y_2(t)$ *solve?*

ACTIVITY 15.5. *Consider the initial value problem*

$$\frac{d}{dt}Y = \begin{pmatrix} 2 & 3 \\ 7 & 6 \end{pmatrix}Y \qquad Y(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

(1) Verify that

$$Y_1(t) = e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} \quad \textit{and} \quad Y_2(t) = e^{9t}\begin{pmatrix}3\\7\end{pmatrix}$$

are solutions to the differential equation.

(2) Choose α and β so that

$$\alpha Y_1(0) + \beta Y_2(0) = \begin{pmatrix}2\\5\end{pmatrix}.$$

(3) Construct the solution to the IVP.

The previous activity suggest the following program for studying linear systems:

- Given a linear system, we somehow construct two solutions Y_1 and Y_2.
- If $Y_1(0)$ and $Y_2(0)$ are independent then we know that we can use the superposition principle to achieve any initial state.
- This gives us a description of all possible solutions: any solution is a linear combinations of the solutions Y_1 and Y_2. In this case, the generic combination $\alpha Y_1 + \beta Y_2$ is the ***general solution*** to the equation.

Exercises

HW:write-in-matrix-form-1

EXERCISE 15.1. *Re-write the following*

$$\begin{pmatrix}x+y\\x\end{pmatrix}$$

in a matrix form.

HW:write-in-matrix-form-2

EXERCISE 15.2. *Re-write the following*

$$\begin{pmatrix}y\\-x\end{pmatrix}$$

in a matrix form.

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EXERCISE 15.3.

(1) *Verify that*

$$Y_1(t) = e^{6t}\begin{pmatrix}2\\5\end{pmatrix} \qquad \textit{and} \qquad Y_2(t) = e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$$

are both solutions of the system

$$\frac{d}{dt}Y = \begin{pmatrix}1 & 2\\5 & 4\end{pmatrix}Y.$$

(2) *By the Superposition Principle you now know infinitely many solutions of the system. What are they?*

(3) *Find a solution with initial condition*

$$Y(0) = \begin{pmatrix}3\\4\end{pmatrix}$$

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EXERCISE 15.4. *Consider the system of equations*

$$\frac{dY}{dt} = \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}Y.$$

(1) *Verify that*

$$Y_1(t) = e^{3t}\begin{pmatrix}1\\1\end{pmatrix} \qquad \textit{and } Y_2(t) = e^{t}\begin{pmatrix}1\\-1\end{pmatrix}$$

are both solutions of this system.

(2) *By the Superposition Principle you now know infinitely many solutions of the system. What are they?*

(3) *Solve the IVP:*

$$\frac{dY}{dt} = \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}Y, \qquad Y(0) = \begin{pmatrix}1\\0\end{pmatrix}.$$