DAY 14

Linearization

ch:linearization

Suppose we have a system of equations

$$\frac{dP}{dt} = f(P,R) \qquad \frac{dR}{dt} = g(P,R) \tag{14.1}$$

linearize:generic

and suppose that (P_*, R_*) is an equilibrium point of the system. Our goal is to find another system of equations, related to the system (14.1), such that

- the new system closely approximates (14.1) when (P, R) is close to (P_*, R_*) and
- the new system is simple enough that we can explicitly solve it.

If we can find such an approximating system, then we can use it to determine how solutions to (14.1) behave when they are close by to the equilibrium point. In particular, such an approximating system can be used to understand the stability of the equilibrium point.

The tool we use is that of linear approximation. Recall that for a smooth function $f(x)$ we have

$$f(x) \approx f(x_*) + \frac{df}{dx}(x_*)\,(x - x_*) \qquad \text{when } x \approx x_*. \tag{14.2}$$

Taylor-1D

Example 14.1. *Suppose that* $f(x) = \sqrt{x}$. *If* x *is close to* $x_* = 4$ *then we have*

$$\sqrt{x} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}\,(x-4) = 2 + \frac{1}{4}(x-4).$$

In particular,

$$\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) = 2.025.$$

Compare this to a typical output given by a calculator for $\sqrt{4.1}$, *which is* 2.02484567313166. *The approximation is pretty good!*

The linear approximation (14.2) is, of course, just the first-order Taylor approximation of the function f centered at $x = x_*$. For systems, we need the corresponding Taylor approximation for functions of two variables, which is

$$f(x,y) \approx f(x_*,y_*) + \frac{\partial f}{\partial x}(x_*,y_*)\,(x-x_*) + \frac{\partial f}{\partial y}(x_*,y_*)\,(y-y_*), \qquad (14.3)$$

Taylor-2D

which holds when $(x,y) \approx (x_*,y_*)$.

We now apply (14.3) to the functions f and g appearing in (14.1). Notice that $f(P_*,R_*) = 0$ and $g(P_*,R_*) = 0$ because (P_*,R_*) is an equilibrium point. Thus we have

$$f(P,R) \approx \frac{\partial f}{\partial P}(P_*,R_*)\,(P-P_*) + \frac{\partial f}{\partial R}(P_*,R_*)\,(R-R_*)$$
$$g(P,R) \approx \frac{\partial g}{\partial P}(P_*,R_*)\,(P-P_*) + \frac{\partial g}{\partial R}(P_*,R_*)\,(R-R_*),$$

when $(P,R) \approx (P_*,R_*)$. We also have

$$\frac{dP}{dt} = \frac{d}{dt}\left[P-P_*\right] \qquad \text{and} \qquad \frac{dR}{dt} = \frac{d}{dt}\left[R-R_*\right],$$

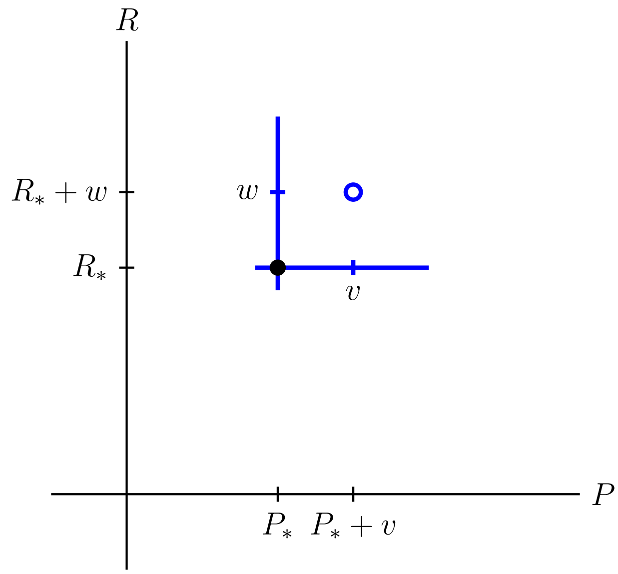
simply because P_* and R_* are constants. Thus when (P,R) are close to the equilibrium point (P_*,R_*) we have the approximate system

$$\frac{d}{dt}\left[P-P_*\right] \approx \frac{\partial f}{\partial P}(P_*,R_*)\,(P-P_*) + \frac{\partial f}{\partial R}(P_*,R_*)\,(R-R_*)$$
$$\frac{d}{dt}\left[R-R_*\right] \approx \frac{\partial g}{\partial P}(P_*,R_*)\,(P-P_*) + \frac{\partial g}{\partial R}(P_*,R_*)\,(R-R_*).$$

We now make a change of variables, setting

$$v = P - P_* \qquad \text{and} \qquad w = R - R_*.$$

The functions v and w describe the displacement of (P,R) away from the equilibrium (P_*,R_*). As the following image depicts, one can view v and w as new coordinates centered at the equilibrium point.



With this change of variables, we obtain the ***linearization*** of (14.1) at (P_*, R_*):

$$\begin{aligned}\frac{dv}{dt} &= \frac{\partial f}{\partial P}(P_*, R_*)\, v + \frac{\partial f}{\partial R}(P_*, R_*)\, w \\ \frac{dw}{dt} &= \frac{\partial g}{\partial P}(P_*, R_*)\, v + \frac{\partial g}{\partial R}(P_*, R_*)\, w.\end{aligned} \tag{14.4}$$

linearize:linearize

The system (14.4) is also called the "linearized equation" at (P_*, R_*).

EXAMPLE 14.2. *Consider the predator-prey system*

$$\frac{dP}{dt} = P(1-P) - PR \qquad \frac{dR}{dt} = -R + 2PR.$$

We linearize this system about the equilibrium point $(P_*, R_*) = (.5, .5)$.

The system takes the form (14.1) *with*

$$f(P, R) = P(1-P) - PR = P - P^2 - PR,$$
$$g(P, R) = -R + 2PR.$$

We can easily compute

$$\frac{\partial f}{\partial P} = 1 - 2P - R \qquad \frac{\partial f}{\partial R} = -P$$
$$\frac{\partial f}{\partial P}(.5, .5) = -.5 \qquad \frac{\partial f}{\partial R}(.5, .5) = -.5$$

and

$$\frac{\partial g}{\partial P} = 2R \qquad \frac{\partial g}{\partial R} = -1 + 2P$$
$$\frac{\partial g}{\partial P}(.5, .5) = 1 \qquad \frac{\partial g}{\partial R}(.5, .5) = 0$$

Thus, setting $v = P - .5$ *and* $w = R - .5$*, we see that the linearized system is*

$$\frac{dv}{dt} = -0.5v - 0.5w$$
$$\frac{dw}{dt} = v$$

Let us pause to summarize our plan for studying equilibrium points for systems of equations.

(1) Find an equilibrium point (P_*, R_*).
(2) Perform a linearization about the equilibrium point to generate a ***linearized equation***, in variables $v = P - P_*$ and $w = R - R_*$, that approximates the original system for v, w small.
(3) Study the linearized equation and obtain a detailed picture of how its solutions behave.
(4) Repeat above steps for all equilibrium points.
(5) Return to the original equation, placing the results about the linearized equations on the larger phase space.

The next several sections are devoted to developing tools for studying the linearized equations.

Exercises

Great-First

Exercise 14.1. *Linearize the following functions in the vicinity of the indicated points:*

(1) $f(x) = x\left(1 - \frac{x}{3}\right)$ *near* $x = 3$*;*
(2) $f(x) = \sqrt{x}$ *near* $x = 4$*;*
(3) $f(x) = \frac{1}{\sqrt{x}}$ *near* $x = 1$*;*
(4) $f(x, y) = x + y + xy$ *near* $x = 1,\ y = 2$*;*
(5) $f(x, y) = x(1 - x) - xy$ *near* $x = 1,\ y = 0$*;*

(6) $f(x,y) = \dfrac{x}{y}$ *near* $x = 2$, $y = 1$;

(7) $f(x,y) = \dfrac{xy}{x+y}$ *near* $x = 1$, $y = 1$.

IceAgeIsComing

EXERCISE 14.2. *Use the ideas of linearization to numerically estimate the following* **without** *the use of 'technology.'* **Then** *'check' your estimates on your calculator.*

(1) $(-0.9)^3$

(2) $\frac{1}{\sqrt{4.01}}$;

(3) $\frac{1}{1.95}$;

(4) $\sqrt[3]{8.09}$

(5) $\sin(0.1)$;

(6) $\ln(0.98)$.

FindThemAll

EXERCISE 14.3. *Find all equilibrium solutions of the following system of differential equations:*

$$\frac{dx}{dt} = 2 + 14x + 12y, \qquad \frac{dy}{dt} = -9 + 12x + 21y.$$

Then linearize the system about each equilibrium.

EqSol-Last

EXERCISE 14.4. *Find all equilibrium solutions of the predator-prey model:*

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy, \qquad \frac{dy}{dt} = 5y\left(1 - \frac{y}{5}\right) - 4xy.$$

Then linearize the system about each equilibrium.

Great-MidBegin

EXERCISE 14.5. *The predator-prey model*

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 5y\left(1 - \frac{y}{5}\right) - 4xy$$

has the co-existence equilibrium of $x = y = 1$.

(1) Linearize the model near this equilibrium.

(2) Generate the vector field plot of the linearized model. What can you say about the stability of the co-existence equilibrium?

Great-Last

EXERCISE 14.6. *Consider the predator-prey model*

$$\frac{dP}{dt} = 0.09P\left(1 - \frac{P}{9}\right) - 2PR$$
$$\frac{dR}{dt} = -0.03R + 0.01PR.$$

(1) *Find all the equilibrium solutions for this model.*
(2) *Linearize the system near each equilibrium, generate the vector field plot of the linearized models, and address the stability of each equilibrium.*

GreatExtra-Only

EXERCISE 14.7. *Consider the predator-prey model*

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{100}\right) - 0.005xy$$
$$\frac{dy}{dt} = \frac{y}{2}\left(1 - \frac{y}{200}\right) + 0.01xy.$$

(1) *Which variable depicts the predator and which variable depicts the prey?*
(2) *What are the equilibrium solutions in this model? Are there any "co-existence" equilibria?*
(3) *Linearize the model near each equilibrium solution.*
(4) *By generating the vector field plot of the linearized models and discuss the stability of each equilibrium solution.*