DAY 13

Nullclines and equilibrium points

In this section we develop a tool for understanding the large-scale behavior of nonlinear systems. Suppose we have a system of the form

$$\frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = g(x, y). \tag{13.1}$$

NL:generic

The motion of solutions in the phase plane is given by the functions f and g. In particular:

- $f(x, y)$ determines the motion in the x direction at location (x, y)
- $g(x, y)$ determined the motion in the y direction at location (x, y).

A ***nullcline*** is a curve in the phase plane where the vector field defined by the differential equation points in a particular direction. For systems of the form (13.1), we focus on two special cases:

Vertical motion nullclines: are locations in the phase plane where

$$\frac{dx}{dt} = 0.$$

This corresponds to points (x, y) such that $f(x, y) = 0$.

Horizontal motion nullclines: are locations in the phase plane where

$$\frac{dy}{dt} = 0.$$

This corresponds to points (x, y) such that $g(x, y) = 0$.

The vertical and horizontal motion nullclines divide the phase plane in to regions. Along the boundary of these regions we know that solutions are either moving horizontally or vertically, depending on which type of nullcline that boundary is. Using the differential equation, we can also determine which direction solutions are moving along various sections of a nullcline. The result is that we understand, at a broad level, the path that solutions may take from one region to the next.

EXAMPLE 13.1. *Consider the system from Example 45.1:*

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \qquad \frac{dy}{dt} = -y + xy.$$

The vertical motion nullclines are given by the condition

$$0 = 2x\left(1 - \frac{x}{2}\right) - xy = x\left(2 - x - y\right).$$

Thus we have two vertical motion nullclines:

$$x = 0 \qquad \textit{and} \qquad 2 - x - y = 0.$$

We first focus on the nullcline $x = 0$, which corresponds to the y axis. Along this nullclines, the vertical motion is described by

$$\frac{dy}{dt} = -y + (0)(y) = -y.$$

Thus we see that the motion of solutions along the y axis is upward when $y < 0$ and downward when $y > 0$. There is no motion when $y = 0$.

From this we deduce the following:

- *Since the motion along the nullcline $x = 0$ is vertical, and the nullcline itself is a vertical line, no solutions can cross this nullcline.*
- *The point $(x, y) = (0, 0)$ must be an equilibrium point, since there is no motion in either x or y directions.*

We now focus on the nullcline $2 - x - y = 0$, which is a straight line in the phase plane. Along this this nullcline, the vertical motion can be deduced by writing the line as $y = 2 - x$ and substituting in to the differential equation for y. Thus the vertical motion is given by

$$\frac{dy}{dt} = -(2 - x) + x(2 - x) = -(x - 2)(x - 1),$$

which is positive when $1 < x < 2$ and is otherwise negative. Thus we see that solutions curves in the phase plane crossing the line $y = 2 - x$ must cross it in the upward direction when $1 < x < 2$ and otherwise *must cross it in the downward direction. Notice that there is no motion at points where $x = 1$ or $x = 0$. Thus we must have equilibrium points at $(x, y) = (1, 1)$ and $(x, y) = (2, 0)$.*

We now turn to the horizontal motion nullclines, which occur when

$$0 = -y + xy = y(x - 1).$$

Thus we have two horizontal motion nullclines, at

$$y = 0 \qquad \text{and} \qquad x = 1.$$

The nullcline $y = 0$ is the x axis. The horizontal motion along this nullcline is given by

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) = x(2 - x).$$

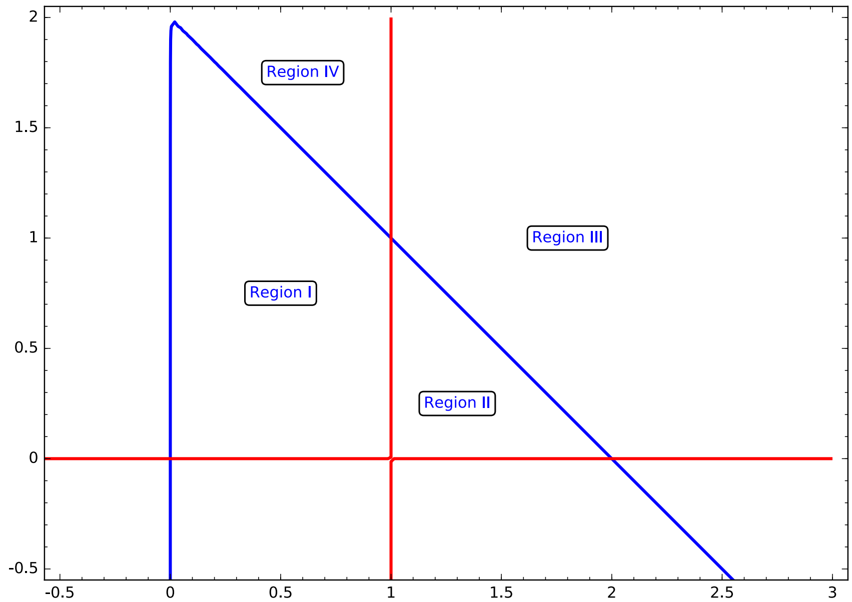
Thus we see that motion is to the right when $0 < x < 2$ and is otherwise to the left. Since the motion along $y = 0$ is horizontal, no solutions can cross this nullcline.

The nullcline $x = 1$ is a vertical line passing through the equilibrium at $(1, 1)$. The horizontal motion along this nullcline is given by

$$\frac{dx}{dt} = 2(1)\left(1 - \frac{1}{2}\right) - (1)y = 1 - y.$$

Thus when $y < 1$ the horizontal motion of solutions in the phase plane along this nullcline is to the right, while when $y > 1$ the horizontal motion of solutions is to the left.

In combination, the four nullclines above divide the phase plane in to a number of regions. Since the system is a population model, we focus on those regions in the first quadrant, labeling them as follows:



We now use this diagram to understand, in a very rough way, the trajectory of typical solutions. First consider Region I. Solutions cannot cross the nullclines $x = 0$ *or* $y = 0$*, and the motion across the boundary between Regions I and IV is downward. Thus a solution in Region I either stays there (perhaps tending towards one of the equilibria) or pass through the horizontal motion nullcline in to Region II.*

Solutions in Region II can either tend to one of the equilibrium points, or pass through the vertical motion nullcline in to Region III.

At first sight, we might tend towards one of the equilibrium points or might tend off to infinity, but if they leave they must do so by crossing the horizontal motion nullcline in to Region IV. In fact, for a solution in Region III we have $y > 2 - x$ *and* $x > 0$*. Thus for such a solution we have* $-xy < -x(2-x)$ *and hence*

$$\frac{dx}{dt} < 2x\left(1 - \frac{x}{2}\right) - x(2-x) = 0.$$

Thus solutions in Region III are always moving to the left, making it more likely that they will cross in to Region IV.

Solutions in Region IV satisfy $0 < x < 1$ *and* $y > 0$. *Thus their vertical motion satisfies*

$$\frac{dy}{dt} = -y + xy < -y + y = 0.$$

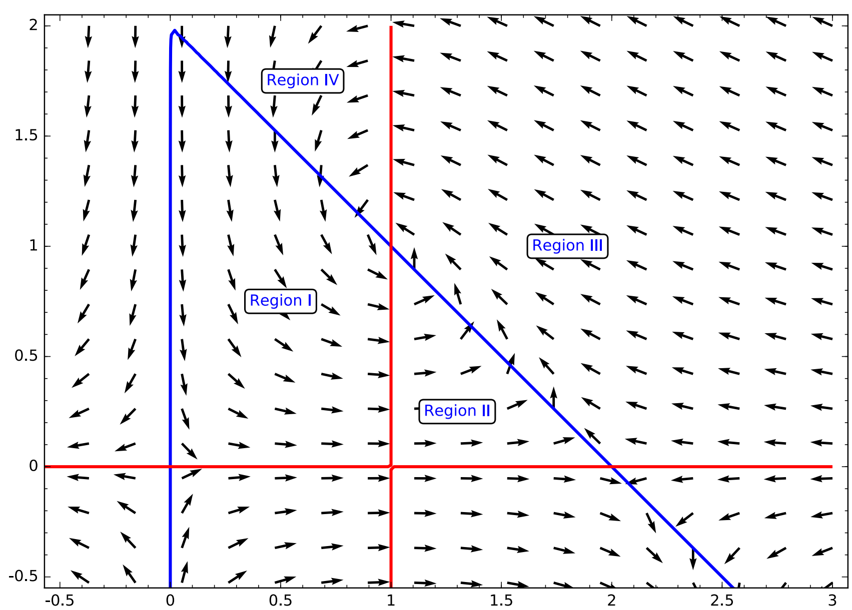
Thus solutions in this region are always moving downward. If they leave Region IV, it must be by crossing the vertical motion nullcline in to Region I.

In summary, it seems that solutions in the first quadrant must do one of the following:

- *tend towards an equilibrium solution,*
- *tend to infinity in the vertical direction in Region III, or*
- *follow a cycle*

Region I $\rightarrow$ *Region II* $\rightarrow$ *Region III* $\rightarrow$ *Region IV* $\rightarrow$ *Region I* $\rightarrow$ *etc.*

In fact, if we superimpose the vector field plot on top of the nullclines, we see that this cyclical behavior is precisely what we expect from solutions!

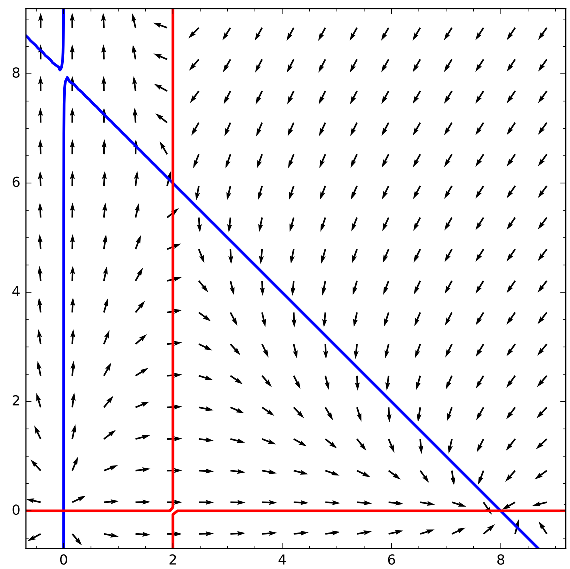


Using nullclines, we can get a good sense of how solutions behave away from equilibrium points. However, the pictures in the previous example don't give us a very good sense of what happens near equilibrium points. Thus we turn to the analysis of equilibrium points next.

Activity 13.1. *Construct the nullclines for the system*

$$\frac{dx}{dt} = 8x\left(1 - \frac{x}{8}\right) - xy \qquad \frac{dy}{dt} = 4y - 2xy.$$

You should get a picture that looks roughly like this:



Finally, here is some Sage code that you can use to generate pictures.

```
var('x,y')

f = 8*x*(1-x/8) - x*y
g = 4*y - 2*x*y

Vertical = implicit_plot(f,(x,-1,9),(y,-1,9),linewidth
    =2,color='blue')
Horizontal = implicit_plot(g,(x,-1,9),(y,-1,9),
    linewidth=2,color='red')

Field = vector((f,g))/vector((f,g)).norm()
Fieldplot = plot_vector_field(Field,(x,-1,9),(y,-1,9))

Mainplot = Fieldplot + Vertical + Horizontal
```

```
Mainplot.show()
```

Null-Only

EXERCISE 13.1. *Consider the system*

$$\frac{dx}{dt} = x - xy \qquad \frac{dy}{dt} = -2y + 2xy$$

(1) Find the equilibrium points of the system.
(2) Find the nullclines of the system.
(3) Draw a phase diagram for the system that includes the nullclines and the equilibrium points.
(4) What can you say about solutions to this system?

NLE-Only

EXERCISE 13.2. *The following system of equations models the populations (in millions) of two competing animal species:*

$$\begin{cases} \dfrac{dx}{dt} = 2x\left(1 - \dfrac{x}{2}\right) - xy \\[2ex] \dfrac{dy}{dt} = 4y\left(1 - \dfrac{y}{4}\right) - 3xy. \end{cases}$$

(1) Find the equilibrium points of the system.
(2) Find the nullclines of the system.
(3) Draw a phase diagram for the system that includes the nullclines and the equilibrium points.
(4) What can you say about solutions to this system?