Writing Assignment 1:
Comparing Harvesting Models

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Abstract

We analyze two models for the harvesting of a yogurt culture growing according to the logistic model: constant-rate harvesting and constant-percentage harvesting. We find conditions that ensure equilibrium populations exist, and analyze their stability. We furthermore determine the harvesting level that yields the maximum absolute harvesting rate. While the maximum absolute harvesting rate is the same for the two models, the stability of the corresponding equilibrium populations is not the same.

1 Introduction

In this paper we consider two harvesting strategies for harvesting yogurt cultures. We assume that the yogurt culture has a constant ideal growth rate r and that the yogurt culture is being grown in an environment with carrying capacity K. The assumption that the relative growth rate is proportional to the available room to grow leads to the logistic equation for the amount of available yogurt Y. Thus in the absence of any harvesting, we assume that Y evolves according to

$$\frac{dY}{dt} = rY\left(1 - \frac{Y}{K}\right). \tag{1}$$

The two harvesting strategies we consider lead to modifications of (1).

The first strategy, which we call *constant-percentage harvesting*, involves the assumption that we are constantly removing h percent of the yogurt. This assumption is described by the differential equation

$$\frac{dY}{dt} = rY\left(1 - \frac{Y}{K}\right) - hY. \tag{2}$$

We assume that $h > 0$.

The second strategy, which we call *constant-rate harvesting*, involves the assumption that we are removing some fixed rate H of yogurt. This assumption is described by the differential equation

$$\frac{dY}{dt} = rY\left(1 - \frac{Y}{K}\right) - H. \tag{3}$$

We assume that $H > 0$.

The two models (2) and (3) have the advantages of being rather simple and capturing the coarsest features of a yogurt-growing operation. But the models do neglect several aspects of growing and harvesting yogurt that occur in actual commercial production. For example, the models assume that the harvesting is taking place continuously rather than at discrete times (as would be the case in a yogurt factory). Nevertheless, if harvesting times are very frequent, then this assumption is a reasonable approximation.

In the following sections we make a qualitative analysis of the models (2) and (3). Our approach to analyzing the models is to find the equilibrium values and to determine their stability. The reason for this is that we expect any commercial operation to function in a state where the population is at (or very nearly at) an equilibrium value.

Following the analyses of the two models, we make a comparision between the two models. Finally, we make some comments regarding how one might interpret our analysis in the context of a commercial yogurt operation.

2 The constant-percentage harvesting model

The first step in our analysis of the constant-percentage harvesting model (2) is to find the equilibrium points. Solving

$$0 = rY\left(1 - \frac{Y}{K}\right) - hY \tag{4}$$

for Y, we find that there are two equilibrium solutions: $Y = 0$ and

$$Y = \frac{r-h}{r}K. \tag{5}$$

From (5) we see that in order to have a positive equilibrium solution the harvesting rate h must satisfy

$$h < r. \tag{6}$$

Assuming that $h < r$, it is easy to verify from (2) that the equilibrium (5) is stable. A typical slope-field plot for the model appears in Figure 1.

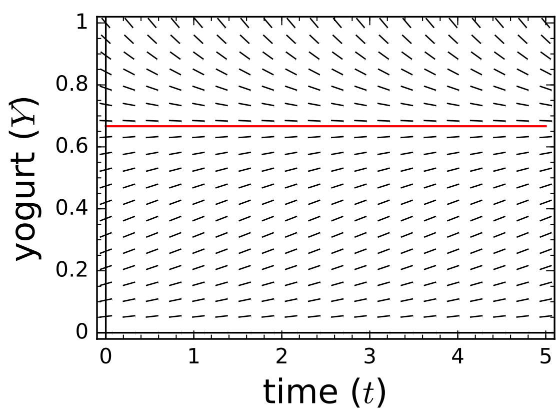

Figure 1: The slope field plot for the percent-rate harvesting model when $h < r$. For the purposes of generating the plot, we assumed that $r = 1$, $K = 1$, and $h = 1/3$. The red line indicates the equilibrium value.

For a given choice of h, the absolute harvesting rate A (which is the rate at which yogurt is being harvested) is given by $A = hP$. At equilibrium, we have

$$A = \frac{h(r-h)}{r}K. \tag{7}$$

If we view A as a function of h, then we see that the maximum value of A occurs when $h = \frac{r}{2}$. For this optimal choice of h, we have that the equilibrium population is

$$Y = \frac{K}{2} \tag{8}$$

and that the absolute harvesting rate is

$$A = \frac{Kr}{4}. \tag{9}$$

This last expression represents the largest possible rate at which we can sustainably harvest yogurt. The slope field for the model in this maximum-harvest scenario is shown in Figure 2.

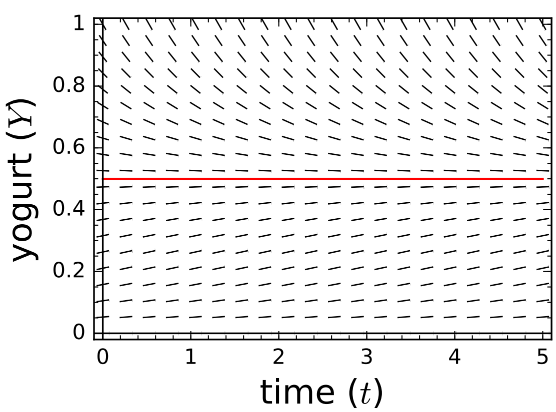

Figure 2: The slope field for the precent-rate harvesting model in the maximum yield scenario, where $h = r/2$. For the purposes of generating the plot we assumed that $r = 1$ and $K = 1$. The red line indicates the equilibrium value at $Y = K/2$.

3 The constant-rate harvesting model

As with the precent-rate model, we begin our analysis of (3) by identifying the equilibrium solutions. Solving

$$0 = rY\left(1 - \frac{Y}{K}\right) - H \tag{10}$$

for Y, we find that there are equilibrium solutions at

$$Y = \frac{K}{2} \pm \frac{K}{2}\sqrt{1 - \frac{4Hr}{K}}. \tag{11}$$

Thus in order for an equilibrium solution to exist we must have

$$1 - \frac{4Hr}{K} \geq 0. \tag{12}$$

It is helpful to rewrite this condition as

$$H \leq \frac{Kr}{4}. \tag{13}$$

In the case that $H < \frac{Kr}{4}$ there are two equilibrium solutions. One can directly verify that the larger equilibrium

$$Y_{+} = \frac{K}{2} + \frac{K}{2}\sqrt{1 - \frac{4Hr}{K}} \tag{14}$$

is stable, while the smaller equilibrium

$$Y_{-} = \frac{K}{2} - \frac{K}{2}\sqrt{1 - \frac{4Hr}{K}} \tag{15}$$

is unstable. The slope field in this case appears in Figure 3.

In the case that $H = \frac{Kr}{4}$, which corresponds to the maximum absolute harvest, there is only one equilibrium solution $Y = \frac{K}{2}$. This equilibrium solution is not stable, as is indicated in Figure 4.

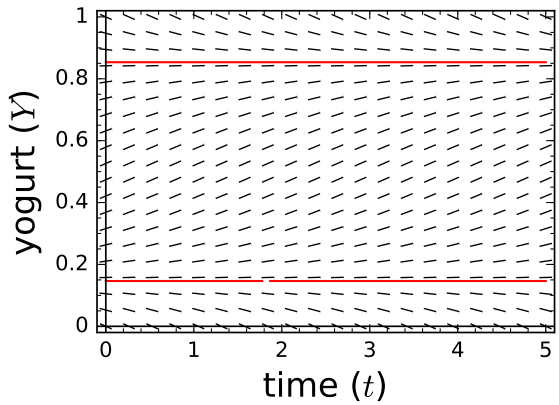

Figure 3: The slope field for the constant-rate harvesting model in the case where $H < \frac{Kr}{4}$. For the purposes of generating the plot we assumed that $r = 1$, $K = 1$, and $H = 1/8$. The red lines indicate the two equilibrium solutions.

4 Discussion

The constant-percentage harvesting model and the constant-rate harvesting model have a number of features in common, but also have a number of features that distinguish them.

For both models we find restrictions on the harvesting that are required in order to have equilibrium solutions. For the constant-percentage model, we have the rather intuitive condition that the harvest rate cannot exceed the ideal growth rate of the yogurt culture itself; see (6). For the constant-rate model, we find less intuitive condition that the rate of harvest cannot exceed one-fourth of the amount that a full habitat would grow under unrestricted conditions; see (13).

In both models we are also able to choose a harvesting strategy that allows us to optimize the absolute harvesting rate. For the constant-percentage model, this is accomplished by choosing the harvest rate to be half the ideal growth rate. Under this choice, there is an equilibrium solution of $K/2$ and the absolute harvest rate is $Kr/4$. Under the constant-rate harvesting model, the maximum absolute harvest is also achieved at a rate of $Kr/4$,

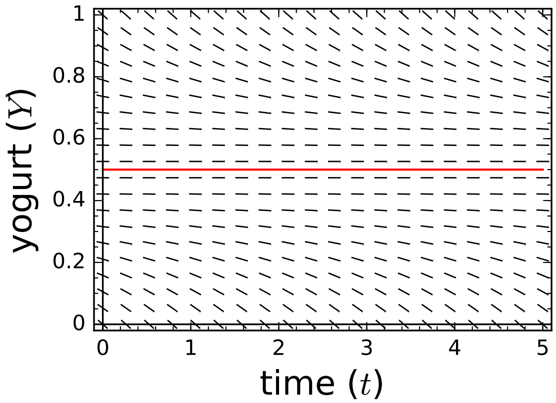

Figure 4: The slope field for the constant-rate harvesting model in the case where $H = \frac{Kr}{4}$. For the purposes of generating the plot we assumed that $r = 1$ and $K = 1$; thus $H = 1/4$. The red line indicates the equilibrium solution.

and the equilibrium solution is also $K/2$ in this situation. It is interesting that, regardless of harvesting strategy, the maximum absolute harvest rate, and corresponding equilibrium population, is the same.

There are two striking differences between the models, however. The first is that the constant-percentage harvesting strategy only yields one non-zero equilibrium solution; this equilibrium is always stable. This is in contrast to the constant-rate model that, for most choices of harvesting rate, has two non-zero equilibrium solutions. The lower of these two non-zero equilibrium solutions is unstable. Thus in the constant-rate model there is always a threshold below which yogurt populations will die off in finite time. This means that yogurt grower using the constant-rate harvesting strategy would need to be careful not to harvest if the yogurt population is below this threshold.

The second difference occurs in the case of optimal harvest. For the constant-percentage strategy, the optimal harvest rate can be achieved at a stable equilibrium. But for the constant-rate strategy, the equilibrium is unstable when the harvest rate is maximized. The result is that a yogurt grower using the constant-rate strategy would need to be very careful.