1.9 Integrating factors

sec:integrating-factors

The method of propagators introduced in the previous section is important in part because it can be generalized to many situations besides those considered in that section; see Section 8.51. However, for simple initial value problems of the form

$$\frac{dy}{dt} = ry + f \quad y(t_0) = y_0, \tag{1.9.1}$$

int:generic-ivp

where both r and f are functions of t, there is a "trick" – called the ***method of integrating factors*** – that is sometimes useful.

The idea of the trick is reverse engineer the product rule. Recall that for two functions u and y says that

$$\frac{d}{dt}[uy] = u\frac{dy}{dt} + \frac{du}{dt}u.$$

In order to apply this to (1.9.1) we write the differential equation as

$$\frac{dy}{dt} - ry = f.$$

If we multiply this by some function u we obtain

$$u\frac{dy}{dt} + (-ru)y = uf. \tag{1.9.2}$$

int:looks-suspicious

The left side of this last equation would look like the derivative of uy if

$$\frac{du}{dt} = -ru. \tag{1.9.3}$$

int:make-this-happen

Let's assume that we have chosen the function u so that it is a solution to (1.9.3). Then (1.9.2) becomes

$$\frac{d}{dt}[uy] = uf.$$

We can integrate this last equation from time t_0 to time t, which leads to

$$u(t)y(t) = u(t_0)y(t_0) + \int_{t_0}^{t} u(\tau)f(\tau)\,d\tau.$$

Dividing by $u(t)$ and using the initial condition $y(t_0) = y_0$ leads to a formula for the solution to (1.9.1), namely

$$y(t) = \frac{u(t_0)}{u(t)}y_0 + \frac{1}{u(t)}\int_{t_0}^{t} u(\tau)f(\tau)\,d\tau. \tag{1.9.4}$$

int:generic-solution

At this stage, you might think that this looks familiar. . . and it is! Basically, it is the same formula as (1.8.8).

Example 1.9.1. *Consider the differential equation*

$$\frac{dy}{dt} + 2y = 7.$$

Multiplying by u, we can rewrite this equation as

$$u\frac{dy}{dt} + y(-2u) = 7u. \tag{1.9.5}$$

int:ex1-rearrange

We want u to satisfy

$$\frac{du}{dt} = -2u,$$

which is easy to arrange by choosing $u = e^{-2t}$. *Thus* (1.9.5) *becomes*

$$\frac{d}{dt}\left[e^{-2t}y\right] = 7e^{-2t}.$$

We can integrate this in order to obtain

$$e^{-2t}y(t) - y(0) = -\frac{7}{2}\left(e^{-2t} - 1\right).$$

Therefore solutions to the differential equation take the form

$$y(t) = e^{2t}y(0) - \frac{7}{2} + \frac{7}{2}e^{2t}.$$

Activity 1.9.1. *Use the method of integrating factors to find formulas for the following differential equations:*

1. $\dfrac{dy}{dt} + 2ty = t^2$
2. $\dfrac{dy}{dt} + y = \cos t$
3. $\dfrac{dy}{dt} + \cos t\, y = \cos t$
4. $\dfrac{dy}{dt} + y = t^2$

Activity 1.9.2. *Suppose a* 1/2 *gallon bottle is being filled with water from a rusty pipe. Water coming out of the pipe at one gallon per minute. As the pipe gets flushed out, there is less rust coming out; we assume that the concentration of rust is* $0.01e^{-5t}$ *ounces per gallon. We furthermore assume that the bottle overflows so that it is always full of water and that there is perfect mixing. If there is* 0.05 *ounces of rust in the bottle, find a formula for how much rust is in the bottle as a function of time. At what time is there only* 0.0001 *ounces of rust?*

Exercise 1.9.1.

SLEN-First

Find the general solution of the following equations:

1. $\dfrac{dy}{dt} = y + t^2$

2. $\dfrac{dy}{dt} = -2ty + e^{-t^2}$

3. $\dfrac{dy}{dt} = -\dfrac{y}{t} + \sin(t)$

SgtPepper

Exercise 1.9.2. Solve the following initial value problems:

1. $\dfrac{dy}{dt} = y + \sin(t), \quad y(0) = 1$

2. $\dfrac{dy}{dt} = \dfrac{y}{t} + t, \quad y(1) = 0.$

Tolstoy

Exercise 1.9.3. A 100-gallon mixing tank is full of pure water at time $t = 0$. Salty water of salt concentration 0.4 lb/gal is being pumped into the tank at a decreasing rate of $e^{-0.05t}$ gal/min. The resulting salty water is also being drained from the tank so that its volume is kept constant at 100 gallons. Assuming the tank is always thoroughly mixed, find the amount of salt in the tank (in pounds) at time t. What will the concentration of salt roughly become in the long run?

SLEN-Last

Exercise 1.9.4. A 1 000-gallon tank is full of pure water. Salty water is being pumped into the tank at a decreasing rate of $\frac{1000}{10+t}$ gallons per hour; here the variable t denotes the number of hours since the beginning of the mixing process. The concentration of salt in the solution which is pumped into the tank is 0.01 pounds per gallon. The tank is constantly being mixed and drained so that the volume of the tank is maintained at 1 000 gallons. How much salt will there be in the tank in the long run?