1.8 Propagator functions

sec:propagator-functions

In this section we consider differential equations of the form

$$\frac{dy}{dt} = ry + f, \tag{1.8.1}$$

prop:generic

where both r and f are functions of t. We interpret (1.8.1) as a growth model with time-dependent relative growth rate r and forcing term f.

Example 1.8.1. *Suppose an investment is growing with variable annual interest rate $r(t) = 0.02 + 0.15\cos(t)$. Suppose furthermore that additional funds are added to the principle investment according to the function $f(t) = e^{-3t}$, measured in thousands of dollars per year. Let $V(t)$ be the value of the investment fund, measured in thousands of dollars, at time t, measured in years. Then growth of V is described by*

$$\frac{dV}{dt} = (0.02 + 0.15\cos(t))V + e^{-3t}.$$

The differential equation (1.8.1) is not separable. Nevertheless, we are able to obtain reasonable formulas for solutions. To accomplish this, we first consider the initial value problem in the special case that there is no forcing, namely:

$$\frac{dy}{dt} = ry \quad y(t_0) = y_0. \tag{1.8.2}$$

prop:special-ivp

The differential equation in (1.8.2) *is* separable, and thus can be addressed using the methods of the previous section.

However, we can also arrive at an understanding of the formula for the solution to (1.8.2) by the following line of reasoning. First, in the case that r is constant, the solution to (1.8.2) is

$$y(t) = y_0 \exp\left(r(t - t_0)\right).$$

We rewrite this as

$$y(t) = y_0 \exp\left(\int_{t_0}^{t} r\, d\tau\right). \tag{1.8.3}$$

prop:special-ivp-integral

Suppose now that r is in fact a function. In the case the function given by (1.8.3) is

$$y(t) = y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right). \tag{1.8.4}$$

prop:special-ivp-solution

We can directly compute, using the Fundamental Theorem of Calculus and the chain rule, that

$$\begin{aligned} y'(t) &= \frac{d}{dt}\left[y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau \right) \right) \\ &= y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau \right) \frac{d}{dt} \int_{t_0}^{t} r(\tau)\, d\tau \\ &= y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau \right) r(t) \\ &= r(t) y(t). \end{aligned} \tag{1.8.5}$$

Furthemore

$$y(t_0) = y_0 \exp\left(\int_{t_0}^{t_0} r(\tau)\, d\tau \right) = y_0 \exp(0) = y_0.$$

Thus the function $y(t)$ determined by (1.8.4) is the solution to (1.8.2).

Notice the structure of the formula (1.8.4) – it simply consists of the initial value y_0 multiplied by some function. If we define the function $P(t, s)$ by

$$P(t, s) = \exp\left(\int_{s}^{t} r(\tau)\, d\tau \right), \tag{1.8.6}$$

prop:prop

then the solution to (1.8.2) is given by $y(t) = P(t, t_0) y_0$.

The function $P(t, s)$ given by the formula (1.8.6) is called the ***propagator function*** for the rate function $r(t)$. Multiplication by $P(t, s)$ corresponds to propagation according to the differential equation in (1.8.2), starting at time s and ending at time t.

Propagator functions have the following properties:

1. $P(s, s) = 1$,
2. $\frac{d}{dt} P(t, s) = r(t) P(t, s)$,
3. $P(t, s) = P(s, t)^{-1}$,
4. $P(t_2, t_1) P(t_1, t_0) = P(t_2, t_0)$.

The first two properties imply that $P(t, s)$ is the unique solution to the initial value problem

$$\frac{dy}{dt} = r(t) y \qquad y(s) = 1. \tag{1.8.7}$$

We can also interpret the first property as the statement that propagating for a time interval of zero size does not change the value of y. The third property states that propagation from time s to time t is the opposite (in the multiplicative sense) of propagation from time t to time s. The fourth property states that propagation from time t_0 to time t_1 and then from time t_1 to time t_2 is the same as propagation from time t_0 to time t_2.

We now show how to use propagator functions in order to find an expression for the solution to the forced equation (1.8.1) with initial condition $y(t_0) = y_0$. First we develop a formula using a intuitive, non-rigorous reasoning; then we verify that the formula solves the desired initial value problem. We begin by supposing that the value of y at time t will be the sum of two terms, one due to the influence of the initial condition and one due to the influence of the forcing. The term due to the initial condition we suppose to be $P(t, t_0)y_0$ (just as in the case when there is no forcing).

The forcing term can be viewed as continuously contributing an "infinitesimal additional amount" to the system. In particular, at a given time s, the quantity $f(s)\,ds$ represents the amount added to the system at that time. (Note that $f(s)\,ds$ has the same units as y.) In order to compute the impact that $f(s)\,ds$ has on the value of $y(t)$ we use the propagator function; the contribution is $P(t, s)f(s)\,ds$. Finally we "add up" all of these infinitesimal contributions, starting with those at time t_0 and ending with those at time t. The result is that we expect the contribution to the value of $y(t)$ due to forcing to be

$$\int_{t_0}^{t} P(t, s)f(s)\,ds$$

and thus we conjecture that the solution to (1.8.1) with initial condition $y(t_0) = y_0$ is

$$y(t) = P(t, t_0)y_0 + \int_{t_0}^{t} P(t, s)f(s)\,ds. \tag{1.8.8}$$

prop:generic-solution

The formula (1.8.8) is called ***Duhamel's formula***.

Let's now verify that the function given by (1.8.8) does indeed solve both (1.8.1) and satisfy the initial condition $y(t_0) = y_0$. The initial condition is straightforward to see, so we focus on the differential equation. It is convenient to use the identity

$$P(t, s) = P(t, t_0)P(t_0, s)$$

in order to rewrite (1.8.8) as

$$y(t) = P(t, t_0)y_0 + P(t, t_0)\int_{t_0}^{t} P(t_0, s)f(s)\,ds.$$

We compute using the properties of propagator functions that

$$\begin{aligned}
y'(t) &= \frac{d}{dt}\left[P(t,t_0)y_0 + P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds\right] \\
&= \frac{d}{dt}\left[P(t,t_0)\right]y_0 \\
&\qquad + \frac{d}{dt}\left[P(t,t_0)\right]\int_{t_0}^{t} P(t_0,s)f(s)\,ds + P(t,t_0)\frac{d}{dt}\int_{t_0}^{t} P(t_0,s)f(s)\,ds \\
&= r(t)P(t,t_0)y_0 + r(t)P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds + P(t,t_0)P(t_0,t)f(t) \\
&= r(t)\left(P(t,t_0)y_0 + P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds\right) + (1)f(t) \\
&= r(t)y(t) + f(t).
\end{aligned}$$

Thus we see that the formula (1.8.8) does indeed satisfy the IVP.

One interesting and important feature of (1.8.8) is that we see explicitly that the solution $y(t)$ is the sum of two parts: the ***homogeneous solution***

$$P(t,t_0)y_0,$$

which due to the initial condition y_0, and the ***particular solution***

$$\int_{t_0}^{t} P(t,s)f(s)\,ds$$

which due to the forcing. Later in the course we see encounter this same phenomenon.

Example 1.8.2. *Consider the initial value problem*

$$\frac{dy}{dt} = t^2 y \qquad y(0) = 17.$$

(Note that this equation is separable; see Exercise **??***.) In this case we have* $r(t) = t^2$*,* $f(t) = 0$*,* $t_0 = 0$*, and* $y_0 = 17$*. We compute the propagator function to be*

$$P(t,s) = \exp\left(\int_s^t \tau^2\,d\tau\right) = \exp\left(\frac{1}{3}t^3 - \frac{1}{3}s^3\right). \tag{1.8.9}$$

Thus the solution is

$$\begin{aligned}
y(t) &= \exp\left(\frac{1}{3}t^3\right)y_0 + \int_0^t \exp\left(\frac{1}{3}t^3 - \frac{1}{3}s^3\right)(0)\,ds \\
&= e^{t^3/3}y_0.
\end{aligned}$$

Example 1.8.3. *Let's use propagators to solve the initial value problem*

$$\frac{dy}{dt} = \cos t - 7 \qquad y(0) = \sqrt{17}.$$

In this IVP the rate function is $r(t) = -7$. Thus the propagator is given by

$$P(t,s) = \exp\left(\int_s^t -7d\tau\right) = e^{-7(t-s)}.$$

The initial value $y_0 = \sqrt{17}$ is specified at time $t_0 = 0$. Thus the homogeneous solution is

$$P(t,0)y_0 = \sqrt{17}e^{-7t}.$$

The forcing function is $f(t) = \cos t$; hence the particular solution is

$$\int_0^t P(t,s)f(s)ds = \int_0^t e^{-7(t-s)}\cos(s)\,ds = \frac{7}{50}\cos(t) - \frac{7}{50}e^{(-7t)} + \frac{1}{50}\sin(t).$$

Thus the solution to the initial value problem is

$$y(t) = \sqrt{17}e^{-7t} + \frac{7}{50}\cos(t) - \frac{7}{50}e^{(-7t)} + \frac{1}{50}\sin(t).$$

Activity 1.8.1. *Use propagators to solve the initial value problem*

$$\frac{dy}{dt} = 1 - \frac{y}{1-t} \qquad y(0) = 31.$$

HW:special-prop-ivp

Exercise 1.8.1. We arrived at the formula (1.8.4) by "intuiting" what the solution "ought" to be. A more rigorous way to proceed would be to notice that the differential equation in (1.8.2) is separable. Use this fact to directly obtain (1.8.4) by the method of separation of variables.

CallMeTheBreeze

Exercise 1.8.2. Verify (by direct computation) the properties of the propagator function $P(t,s)$:

1. $P(s,s) = 1$,
2. $\frac{d}{dt}P(t,s) = r(t)P(t,s)$,
3. $P(t,s) = P(s,t)^{-1}$,
4. $P(t_2,t_1)P(t_1,t_0) = P(t_2,t_0)$.

I-keep-running-down-the-road

Exercise 1.8.3. Solve the following IVP using the propagator method:

$$\frac{dy}{dt} = 2y - t \qquad y(0) = 1.$$

SittinByTheDockOfTheBay

Exercise 1.8.4. Solve the following IVP using the propagator method:

$$\frac{dy}{dt} = \frac{y}{1+2t} - 7 \qquad y(0) = 1.$$

ShareItFairly

Exercise 1.8.5. Suppose money is invested in a volatile market that has an annual growth rate of $r(t) = 0.01 + 0.05\cos 10t$, where t is measured in years.

1. Make a plot of $r(t)$ over a 10 year time period. How should one interpret this growth rate?
2. Suppose there is an initial investment of \$100. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?
3. Suppose instead that no money is initially invested, but that one continuously adds to an investment at a rate of \$10 per year. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?

AgainAndOver

Exercise 1.8.6. Solve the initial value problem from Exercise 1.2.4:

> We consider a large urn of coffee in cafeteria. A student is draining coffee in to her thermos while at the same time the brewing machine is adding fresh coffee. We want to keep track of the amount of caffeine (in milligrams) in the urn during this process.
>
> Suppose the following:
>
> - There is initially 4 liters of coffee in the urn, with a caffeine concentration of 100 mg per liter.
> - Starting at time $t = 0$ the brewing machine is adding extra strong coffee, which has a concentration of 250 mg per liter. This new coffee is being added at a rate of 25 mL per minute.
> - Starting at time $t = 0$ the student begins draining coffee out of the urn at a rate of 30 mL per minute.

Your goal is to find a formula for the amount of caffeine in the urn as a function of time.