1.7 Separable equations

review-separable-ode

We now study a class of first-order equations for which it is possible to obtain a formula of sort for solutions. Consider differential equations of the form

$$\frac{dy}{dt} = f(t, y). \tag{1.7.1}$$

sep:generic-generic

If the function $f(t, y)$ appearing in (1.7.1) is the product of a function of y and a function of t, then we say that the differential equation is ***separable***. In particular, separable equations take the form

$$\frac{dy}{dt} = h(y)\, g(t) \tag{1.7.2}$$

sep:generic

for some functions h and g.

Example 1.7.1. *The differential equation*

$$\frac{dy}{dt} = e^{3y} \cos(t)$$

is separable. The equation takes the form (1.7.2) *with* $h(y) = e^{3y}$ *and* $g(t) = \cos(t)$.

Separable equations are special because it is possible to obtain an implicit formula for their solutions. To see this, we divide (1.7.2) by $f(y)$ and also change notation in order to emphasize the fact that y is a function of t. The result is that (1.7.2) is equivalent to

$$\frac{1}{h(y(t))}\, y'(t) = g(t).$$

We now integrate from time t_0 to a generic time t, obtaining the formula

$$\int_{t_0}^{t} \frac{1}{h(y(\tau))}\, y'(\tau)\, d\tau = \int_{t_0}^{t} g(\tau)\, d\tau. \tag{1.7.3}$$

sep:generic-intermediate

(Note that since we are using t as one of the limits of integration, we must use another letter as the variable of integration; we choose the Greek letter τ.) Finally, we perform a change of variables on the left side of (1.7.3) in order to arrive at the following formula:

$$\int_{y(t_0)}^{y(t)} \frac{1}{h(y)}\, dy = \int_{t_0}^{t} g(\tau)\, d\tau. \tag{1.7.4}$$

sep:theory

The equation (1.7.4) provides an implicit formula for the solution $y(t)$ to the differential equation (1.7.2). In situations where the functions $h(y)$ and $g(t)$ are relatively simple, we can use (1.7.4) in order to obtain explicit formulas for $y(t)$.

ex:sep-1

Example 1.7.2. *Consider the differential equation*

$$\frac{dy}{dt} = e^{3y}\cos t$$

appearing in the example above. This equation is equivalent to

$$\int_{y(t_0)}^{y(t)} e^{-3y}\,dy = \int_0^t \cos(\tau)\,d\tau.$$

Integrating we find

$$-\frac{1}{3}e^{-3y(t)} + \frac{1}{3}e^{-3y(t_0)} = \sin(t) - \sin(t_0).$$

Rearranging algebraically, we find that

$$y(t) = -\frac{1}{3}\ln\left(e^{-3y(t_0)} - 3\sin(t) + 3\sin(t_0)\right).$$

Notice that the constants t_0 and $y(t_0)$ in (1.7.4) can be chosen arbitrarily, so long as $h(y(t_0))$, $g(t_0)$ are defined and $h(y(t_0))$ is not zero. These two constants reflect the freedom of choosing initial conditions in the initial value problem associated to (1.7.1). If we have an initial condition $y(t_0) = y_0$, then we can replace t_0 and $y(t_0)$ accordingly in order to obtain an expression for the unique solution.

If we do not have specific initial conditions that we are interested in, it is often convenient to slightly abuse notation and write (1.7.4) as the indefinite integral

$$\int \frac{1}{h(y)}\,dy = \int g(t)\,dt. \tag{1.7.5}$$

sep:abuse

(Since we are not using t as one of the limits of integration, we do not need to use the variable τ.) When making use of (1.7.5), it is important to keep in mind that there will be a free constant of integration – if we do have initial conditions, then those conditions determine the constant.

ex:sep-abuse

Example 1.7.3. *Consider again the equation*

$$\frac{dy}{dt} = e^{3y}\cos(t). \tag{1.7.6}$$

sep:abuse-example

The integral corresponding to (1.7.5) *is*

$$\int e^{-3y}\,dy = \int \cos(t)\,dt,$$

which can be integrated to yield

$$-\frac{1}{3}e^{-3y} = \sin(t) + C,$$

where C is some arbitrary constant.

If we are not concerned about satisfying the initial condition, then we may simply solve for y, obtaining the formula

$$y = -\frac{1}{3}\ln\left(-3\sin(t) - 3C\right)$$

for a generic solution.

Note that since C is an arbitrary constant, the quantity $-3C$ is also an arbitrary constant. It is common practice to replace $-3C$ by C, while taking mental note that the two constants are not the same. Thus we may write a generic solution to (1.7.6) as

$$y(t) = -\frac{1}{3}\ln\left(-3\sin(t) + C\right).$$

The following example illustrates that, with proper care in interpretation, the integrals (1.7.4) and (1.7.5) lead to the same result.

Example 1.7.4. *Consider the initial value problem*

$$\frac{dy}{dt} = e^{3y}\cos t \qquad y(0) = 1.$$

Thus $t_0 = 0$ and $y(t_0) = 1$. Using the work of Example 1.7.2, we see that (1.7.4) leads to the formula

$$y(t) = -\frac{1}{3}\ln\left(e^{-3} - 3\sin(t)\right). \tag{1.7.7}$$

sep:ex-result

On the other hand, if we use (1.7.5) we obtain, using the work of Example 1.7.3 we have

$$y(t) = -\frac{1}{3}\ln\left(-3\sin(t) + C\right). \tag{1.7.8}$$

sep:abuse-result

Inserting the initial condition in to this expression we have

$$1 = -\frac{1}{3}\ln\left(-3\sin(0) + C\right),$$

from which we conclude that the constant C must be

$$C = e^{-3}.$$

Inserting this in to (1.7.8) yields the same formula for $y(t)$ as in (1.7.7).

Activity 1.7.1. *Use the integral (1.7.5) in order to find a formula for generic solutions to*

$$\frac{dy}{dt} = (1 + y^2)(1 + t^2).$$

Activity 1.7.2. *Use the integral (1.7.5) in order to find a formula for generic solutions to*

$$\frac{dy}{dt} = (1 - y^2)(1 + t^2).$$

Activity 1.7.3. *Use the integral* (1.7.5) *in order to find a formula for generic solutions to*

$$\frac{dy}{dt} = \frac{ty}{1+t^2}.$$

An important example of a separable equation is the logistic population model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \tag{1.7.9}$$

sep:logistic

where both r and K are positive. When analyzing with this equation, it is helpful to work using the dimensionless variable

$$y(t) = \frac{P(t)}{K},$$

which measures the population $P(t)$ as a percent of the carrying capacity K. Dividing (1.7.9) by K, we replace each instance of P/K by y in order to obtain

$$\frac{dy}{dt} = ry(1-y). \tag{1.7.10}$$

sep:generic-logistic

The integral (1.7.5) for this equation is

$$\int \frac{1}{y(1-y)}\, dy = \int r\, dt. \tag{1.7.11}$$

sep:logistic-integral

In the discussion below, we move freely between the variables P and y.

In preparation for integration, we use partial fractions to write

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Thus the integral becomes

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int r\, dt.$$

Integrating we find

$$\ln\left(\frac{y}{1-y}\right) = rt + C.$$

We apply the exponential to both sides and replace e^C by C, obtaining

$$\frac{y}{1-y} = Ce^{rt}. \tag{1.7.12}$$

sep:logistic-intermediate

Solving algebraically for y we find the following formula for y:

$$y(t) = \frac{Ce^{rt}}{1+Ce^{rt}}. \tag{1.7.13}$$

sep:logistic-reduced-solution

In terms of the variable P, this corresponds to

$$P(t) = \frac{Ce^{rt}}{1 + Ce^{rt}} K.$$

We can interpret the constant C by evaluating (1.7.12) at $t = 0$ to find

$$C = \frac{y_0}{1 - y_0};$$

here

$$y_0 = y(0) = \frac{P(0)}{K}$$

is the percent of the habitat that is initially full and

$$1 - y_0 = 1 - \frac{P(0)}{K}$$

is the precent of the habitat that is initially empty. Thus the constant C represents the ratio of these two percentages. We consider two "generic" cases and three special cases.

The first generic case is when $0 < P(0) < K$, which we interpret at the statement that initially the habitat has a population that is less than the carrying capacity. We call this the "underpopulated" case. In terms of y, this corresponds to $0 < y_0 < 1$, from which we deduce that the constant C is positive. We can compute from (1.7.13) that $y(t)$ exists for all $t \geq 0$ and that

$$\lim_{t \to \infty} y(t) = 1.$$

This corresponds to

$$\lim_{t \to \infty} P(t) = K,$$

which we interpret as the statement that in the long run, the population increases to fill the habitat. Note that mathematically we never have $P(t) = K$ (why not?). However, the difference between $P(t)$ and K eventually becomes less than one individual, in which case the two are "equal for all practical purposes."

Activity 1.7.4. *The second generic situation we consider is when $P(0) > K$, which we interpret as the statement that initially the population exceeds that which the habitat can sustainably support. What are possible values of C in this case? [Hint: It might be helpful to plot C versus y_0.] What is the long-term behavior of $P(t)$ in this case?*

Next, we consider the special case when $P(0) = K$, which corresponds to $y_0 = 1$. Notice that the formula for C does not make sense in this case – in fact, the integral (1.7.11) does not make sense when $y = 1$! However, notice that $y(0) = 1$ is an equilibrium solution to (1.7.10), corresponding to the equilibrium solution $P(t) = K$ to (1.7.9).

Activity 1.7.5. *Consider the special when $P(0) = 0$. What are y_0 and C in this case? What is the corresponding solution $y(t)$? Interpret this solution.*

Finally, we consider the case when $P(0) < 0$. While this case is not physically relevant for population modeling, it is interesting from a mathematical point of view. In this situation, we have $-1 < C < 0$. Using (1.7.13) we see that there is some time $t_* > 0$ such that

$$\lim_{t \to t_*^-} y(t) = -\infty,$$

meaning that solutions blow up in finite time.

We conclude this section by analyzing a mixing problem using separation of variables.

Example 1.7.5. *Suppose we have a 100 gallon tank full of some liquid, in to which a salt water solution is flowing at a rate of 5 gallons per minute. The salt water solution contains 8 ounces of salt per gallon. The salt solution mixes perfectly with the contents of the tank; the resulting mixture is leaving the tank at a rate of 5 gallons per minute.*

Let $S(t)$ be the amount of salt, measured in ounces, in the tank at time t, which we measure in minutes. The differential equation describing this situation is

$$\frac{dS}{dt} = 40 - \frac{1}{20} S.$$

This is a separable equation; the resulting indefinite integral is

$$\int \frac{1}{800 - S}\, dS = \int \frac{1}{20}\, dt.$$

(Note: You don't need to factor out the 1/20, but it is helpful.) Evaluating the integral we find that

$$-\ln(800 - S) = \frac{1}{20} t + C,$$

which we rewrite as

$$S(t) = 800 - Ce^{-t/20}.$$

From this it is easy to see that the solution $S(t)$ exists for all time t and that

$$\lim_{t \to \infty} S(t) = 800.$$

Thus for large times, the amount of salt in the tank approaches 800 ounces.

Notice that we never needed to know what the value of C was! What extra information would we need in order to compute C?

Sep-First

Exercise 1.7.1. Find the general solution of the following equations:

1. $\dfrac{dy}{dt} = y^3$;

2. $\dfrac{dy}{dt} = (y-1)^2$;

3. $\dfrac{dy}{dt} = e^{-t}(2-y)$;

4. $\dfrac{dy}{dt} = y(3-y)$;

5. $\dfrac{dy}{dt} = y(3-y)-2$.

Sep-MidBegin

Exercise 1.7.2. A person makes an initial payment of \$3 000 to a retirement fund, and plans to contribute \$6 400 each year continuously for 40 years until the person is 65. The fund will earn 8% a year.

1. Find an expression for the value of the fund t years after the initial payment was made;

2. What is the value of the fund at age 65;

3. Assume that no payments are going to be made after the age 65 and that the same amount of money is going to be taken from the account each month. How big can this sum of money be in order for the retirement fund to last another 30 years?

FishUnlimited

Exercise 1.7.3. Assuming unlimited resources some particular species of fish grows at a steady per capita rate of 200% per year. However, it is believed that the lake in which this species lives can only support up to 10 000 fish. The lake is private and the owners would like to (continually throughout the year) harvest about 4 800 fish. The current fish population in the lake is about 4 200. Find an expression for the number of fish as a function of time. Graph your solution. Would you agree that the owner is “overharvesting”?

Sep-Last

Exercise 1.7.4. A 100-gallon mixing vat is initially full of brine in which the concentration of salt is $0.5\,\frac{\text{lb}}{\text{gall}}$. Pure water is pumped into the tank at the rate of $3\frac{\text{gall}}{\text{min}}$. Simultaneously 3 gallons of brine per minute are being pumped out. The vat is kept thoroughly mixed at all times.

1. Find the expression for the amount (in lb’s) of salt in the vat after t minutes;

2. When will the concentration of salt drop below $0.1\,\frac{\text{lb}}{\text{gall}}$?