1.3 Solutions to differential equations

Activity 1.3.1. *Find all solutions to the system of equations*

$$x^2 + xy + x = 0,$$
$$-y^2 + xy + 2y = 0.$$

Be systematic... there should be four solutions!

The previous activity involves finding solutions to an *algebraic* equation. It's fairly clear what we mean by a solution: the pair $(-1, 0)$ is a solution because when we plug in -2 for x and 0 for y, the system of equations is satisfied. In other words, a solution is something that "if we plug it in, it works."

The same principle of "plug it in and it works" applies to differential equations as well.

Example 1.3.1. *Consider the differential equation*

$$\frac{dy}{dt} = 5y. \tag{1.3.1}$$

what-is-solution-1

Here the unknown is the function $y(t)$.

It is pretty easy to verify that the function $y(t) = 8e^{5t}$ is a solution to (1.3.1). If we plug this function in to the left side, we get

$$\frac{d}{dt}\left[8e^{5t}\right] = 40e^{t},$$

while if we plug it in to the right side we get

$$5\left[8e^{5t}\right] = 40e^{t}.$$

Since these two are the same, the function $y(t) = 8e^{5t}$ is a solution to (1.3.1).

Notice that there are in fact an infinite number of solutions to (1.3.1). The function $y(t) = Ce^{5t}$ is a solution for any value of C.

act:verify-solution-2

Activity 1.3.2. *Consider the differential equation*

$$\frac{dy}{dt} = 6y + 3t.$$

Show that the function

$$y(t) = e^{6t} - \frac{1}{12} - \frac{1}{2}t.$$

is a solution.

Unfortunately, the principle of "plug it in and it works" does not work for all equations – this is the case both for algebraic equations and for differential equations.

Example 1.3.2. *Consider the algebraic equation*

$$e^x + x = 0. \tag{1.3.2}$$

indirect-alg

No matter how we try, we cannot "isolate x" and find an closed-form expression for a solution to this equation.

However, it is straightforward to deduce that (1.3.2) does indeed have a solution. To see this, consider the function $f(x) = e^x + x$. It is easy to see that

$$f(-1) = e^{-1} - 1 < 0 \quad \text{and} \quad f(0) = e^0 = 1 > 0.$$

Thus by the intermediate value theorem, there must exist a number x_* between -1 and 0 where $f(x_*) = 0$. Clearly the number x_* is a solution to (1.3.2).

Notice that not only have we deduced indirectly that (1.3.2) has a solution, we also have shown one property of the solution, that the solution is between -1 and 0. We can easily deduce another property: since $f'(x) = e^x + 1 > 0$ we see that the function f is strictly increasing and thus there can be only one solution to (1.3.2).

Finally, we can use technology to obtain a numerical approximation of the solution. For example, we can use the following Sage code.

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find_root(exp(x)+x==0,-1,0)
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The previous example illustrates that for algebraic equations, we may not always be able to find nice formulas for solutions to algebraic equations. Nevertheless:

- If we are clever we can deduce whether or not solutions exist.
- In the case that there are solutions, we can also try to deduce how many solutions there are, as well as some properties of those solutions.
- Finally, we can use technology to obtain numerical approximations of solutions.

These three principles also apply to differential equations. In the next several sections we develop some theory that can be used to deduce when differential equations admit solutions (and how many solutions there are), to deduce some properties of solutions to certain classes differential equations, and some techniques for obtaining numerical approximations of solutions to differential equations. Then at the end of this chapter we consider several categories of differential equations for which we can write down explicit formulas for solutions.

knowit

We end this section by examining three examples of differential equations for which we are able to find relatively simple formulas for solutions. They are

$$\frac{dy}{dt} = r, \tag{1.3.3a}$$

knowit-1

$$\frac{dy}{dt} = ry, \tag{1.3.3b}$$

knowit-2

$$\frac{dy}{dt} = ry^2; \tag{1.3.3c}$$

knowit-3

in each case r is a numerical constant.

activity:know-it

Activity 1.3.3.

1. *Show that the general solution to* (1.3.3a) *is*

$$y(t) = C + rt.$$

2. *Show that the general solution to* (1.3.3b) *is*

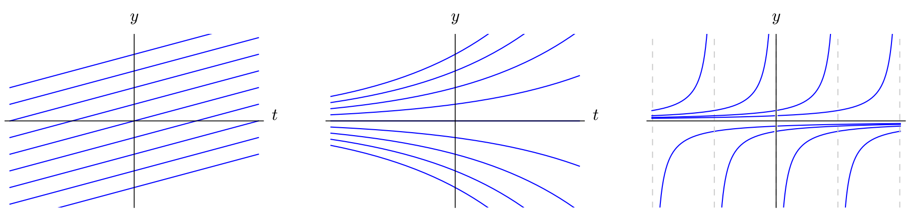
$$y(t) = Ce^{rt}.$$

3. *Show that the general solution to* (1.3.3c) *is*

$$y(t) = \frac{1}{C - rt}.$$

You are strongly encouraged to memorize the solutions presented in the this activity.

We can plot the solutions of the differential equations (1.3.3) for various values of C; see Figure 1.3.1. We notice three things from examining these



(a) Plots of $y(t) = C + rt$. (b) Plots of $y(t) = Ce^{rt}$. (c) Plots of $y(t) = 1/(C - rt)$.

Figure 1.3.1: Plots of the general solutions to (1.3.3a), (1.3.3b), (1.3.3c) for various values of the constant C; these assume that $r > 0$.

knowit-plots

plots:

1. The solution to a differential equation may not be finite for all time – there is the possibility, illustrated by the plots of solutions to (1.3.3c), that the solution will ***blow up*** in finite time t_*, by which we mean $y(t) \to \pm\infty$ as $t \to t_*$. Thus the theory that we develop for differential equations must have two separate aspects: a local-in-time theory indicating that solutions exist and a separate theory regarding what happens as time progresses.

2. The second feature is that for each initial value $y(0)$ at time $t = 0$, there exists a solution. Thus we expect in general that for each "initial condition" there should be one solution to a differential equation.

3. Finally, solutions do not cross in the ty plane, meaning that for each state (t_*, y_*) there exists only one solution that ever achieves that state.

We expect that the theory we develop to incorporate these three points.

YourFirstSolution

Exercise 1.3.1. Determine if the function $y(t) = 1 + 2e^t$ is a solution of the differential equation

$$\frac{dy}{dt} = (y-1)(y-2e^t).$$

TheWord-MidEnd

Exercise 1.3.2.

1. Determine if the function $y(t) = e^t - 2t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.
2. Determine if the function $y(t) = 2e^t - t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.
3. Determine for which values of the constant C the function $y(t) = Ce^t - t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.

DoesItSolveIt

Exercise 1.3.3. Verify that the function $y(t) = \sqrt{\frac{4-t^3}{3t}}$ solves the differential equation

$$\frac{dy}{dt} = -\frac{t^2 + y^2}{2ty}.$$

hw:solve-square-root

Exercise 1.3.4. Consider the differential equation

$$\frac{dy}{dt} = \sqrt{y}.$$

1. Show that for any constant C, the function

   $$y(t) = \left(\frac{1}{2}t + C\right)^2$$

   is a solution (at least when $t \geq -2C$).
2. Show that the function $y(t) = 0$ is also a solution.
3. Find two different solutions to the differential equation that both satisfy $y(0) = 0$.

I-love-Cauchy

Exercise 1.3.5. In this problem we consider a differential equation that involves both first and second derivatives of the unknown function, namely

$$t^2 \frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y = 0.$$

Find those values of α such that the function $y(t) = t^\alpha$ solves the differential equation.

hw:verify-solve-1

Exercise 1.3.6. Verify that for any constant C, the function

$$y(t) = Ce^{6t} - \frac{1}{12} - \frac{1}{2}t$$

satisfies the differential equation

$$\frac{dy}{dt} = 6y + 3t.$$

Use this to find a function y that satisfies both

$$\frac{dy}{dt} = 6y + 3t \quad \text{and} \quad y(0) = 10.$$