# Chapter 1

# First order equations

ch:first-order

chapter:ReadAndWrite

## 1.2 Constructing differential equations

We constructed the differential equation (0.1.1) by assuming that the derivative of the function $P(t)$ was constant. The derivative $P'(t)$ represents the ***absolute growth rate***, also called the ***absolute rate of change***, of the function $P$. Rates of change, however, are often discussed in terms of percentages rather than in absolute terms. Such percentages refer to the ***relative growth rate***, also called the ***percent rate of change***, which is defined by

$$\text{relative growth rate of } P = \frac{P'(t)}{P(t)} = \frac{1}{P}\frac{dP}{dt}.$$

Notice that the units of the relative growth rate are simply 1 divided by the units of $t$; the units of $P$ cancel out.

We can construct differential equations by making assumptions about either the absolute or relative growth rate of a population. The simplest assumption, which is the one made in the previous section, is that the absolute growth rate is constant. The corresponding differential equation is

$$\frac{dP}{dt} = r, \tag{1.2.1}$$

constant-growth-model

where $r$ is some constant. The differential equation (1.2.1) is called the ***constant growth model*** because it describes a population with a constant absolute growth rate.

Alternatively, we may assume that the relative growth rate is constant. The resulting differential equation is

$$\frac{1}{P}\frac{dP}{dt} = r, \tag{1.2.2}$$

basic-growth-model

where $r$ is some constant. The differential equation (1.2.2) is called the ***basic growth model***. The basic growth model describes situations where population $P$ is growing at $r$ percent per unit of time.

Notice that the constant $r$ in (1.2.1) has different units from the constant $r$ appearing in (1.2.2)!

**Example 1.2.1.** *Suppose a fixed amount of money is placed in to a bank account that earns 3% annual interest. Let $A(t)$ be the value of the account at time $t$, where we measure $t$ in years. Then $A$ must satisfy*

$$\frac{1}{A}\frac{dA}{dt} = 0.03.$$

*Notice that the amount of money originally placed in to the account does not affect this differential equation. The equation only describes how the amount is changing, not what the amount actually is!*

Multiplying both sides of (1.2.2) by $P$, we can re-write the basic growth model as

$$\frac{dP}{dt} = rP. \tag{1.2.3}$$

basic-growth-model-2

We can interpret the equation (1.2.3) as the statement that the absolute rate of change of $P$ is proportional to $P$. (Remember that "is proportional to" means "is equal to a constant times.") Thus there are two different assumptions that both lead to basic growth model equation.

It is useful, and interesting, to consider variations on the basic growth model that incorporate additional features.

example:forced-basic

**Example 1.2.2.** *Suppose that the bacteria population in my jar of yogurt has a relative growth rate of 5% per hour. Additionally, I constantly remove yogurt from the jar in such a way that bacteria are being removed at a rate of 7 million per hour.*

*We can construct differential equation modeling this situation as follows. Let $P(t)$ be the population of bacteria in the jar at time $t$, measured in millions; let the time $t$ be measured in hours. We assume that the absolute rate of change of $P$ is the sum of two terms, the first coming from the relative growth rate and the second coming from the removal of the bacteria. The resulting differential equation is*

$$\frac{dP}{dt} = 0.05P - 7. \tag{1.2.4}$$

ex-basic-mod

*The term $0.05P$ comes from the assumption on the relative growth rate of the bacteria, while the term $-7$ comes from the removal of the bacteria. Notice that each of the two terms on the right side of (1.2.4) have units of millions per hour, which is the same as the units of the left side. It is always important that terms being added together have the same units!*

More generally, if population modeled by $P$ grows with relative growth rate $r$ and is subject to "migration" determined by the function $f$, then the $P$ satisfies

$$\frac{dP}{dt} = rP + f. \tag{1.2.5}$$

forced-basic-growth

The function $f$ is sometimes called the ***forcing term***, and thus we can call (1.2.5) the ***forced basic growth model***.

Another important variation on the basic growth model comes from making the assumption that the population being studied lives in a habitat that can only sustain a finite size population, and that the relative growth rate of the population is proportional to the percent of available habitat. In order to construct a differential equation from this assumption, let $P(t)$ describe the size of the population and let $K$ be the population size that the habitat can sustain. The percent of habitat that is available is given by the function

$$1 - \frac{P}{K}.$$

Thus the assumption that the relative growth rate is proportional to the percent of available habitat leads to the differential equation

$$\frac{1}{P}\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right),$$

where $r$ is some constant. We can rewrite this equation as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{1.2.6}$$

logistic-growth-model

The equation (1.2.6) is called the ***logistic growth model***.

Notice that the constant $r$ in the logistic growth model (1.2.6) has the same units as the constant $r$ in the basic growth model (1.2.2). In fact, if $P$ is very small relative to $K$, then the term $P/K$ in (1.2.6) is very close to zero. Thus when the habitat can support a very large population size (relative to the current population size), then the logistic growth model is approximated by the basic growth model. (We make this idea of one differential equation approximating another a bit more precise in the next chapter.) For this reason we say that the constant $r$ in the logistic model is the ***ideal relative growth rate***, or ***relative growth rate under ideal conditions***.

**Example 1.2.3.** *Suppose that the milk in my yogurt jar can sustain a population of* 9 *million bacteria, and that the bacteria has ideal relative growth rate of* 15% *per hour. Then we can model the population of bacteria in the jar by the differential equation*

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{9}\right),$$

*where we measure $P$ in millions and $t$ in hours.*

Notice that in the previous example I chose to measure the population $P$ in millions. We could have decided to measure the population in units of individual bacteria, in which case the differential equation would be

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{9000000}\right).$$

Both of these differential equations describe the same physical situation, but the first one is much easier to work with since the numbers are of more "reasonable size." In general, it is advisable to choose units so that the numbers appearing in the resulting differential equations are as "reasonable" as possible.

It is possible to modify the logistic model in ways analogous to the modification of the basic growth model appearing in Example 1.2.2.

**Activity 1.2.1.** *Construct a differential equation describing the following situation: A population has ideal relative growth rate of* 7% *per year and lives in a habitat that can sustain a population of* 12, 000 *individuals. Furthermore, individuals are moving in to the habitat at a constant rate of* 3, 000 *per year.*

**Activity 1.2.2.** *Construct a differential equation describing the following situation: A population has ideal relative growth rate of* 3% *per month and lives in a habitat that can sustain a population of* 500 *individuals. Furthermore, individuals are continuously leaving the habitat at a rate of* 5% *per month.*

All of the differential equations constructed above are population models. Differential equations can be used to describe a wide variety of phenomena beyond population dynamics. Later in the course we address a number of equations arising in physics, while in the first part of the course we focus primarily on population models. The concept of a "population," however, can encompass a wide range of phenomena, as the following examples demonstrate.

**Example 1.2.4.** *Consider a tank with capacity* 100 *gallons, containing a mixture of fresh water and salt. The contents of the tank are being drained at a rate of* 5 *gallons per minute. Simultaneously, salt water having concentration of* 8 *ounces per gallon is being pumped in to the tank. We assume that the mixture in the tank is being stirred so that the salt concentration in the tank is uniform.*

*We can model this situation by considering the amount of salt in the tank to be a "population." Let* $S(t)$ *be the amount of salt (measured in ounces) in the tank at time* $t$ *(measured in minutes). Then the rate of change of salt in the tank satisfies*

$$\frac{dS}{dt} = (\textit{rate of salt in}) - (\textit{rate of salt out}).$$

*We can easily find expressions for the rate of salt entering and leaving the tank as follows:*

$$\begin{aligned}(\textit{rate of salt in}) &= (\textit{rate of liquid entering tank}) \\ &\qquad \times (\textit{concentration of salt in the incoming liquid}) \\ &= \left(\frac{5 \textit{ gallons}}{\textit{minute}}\right)\left(\frac{8 \textit{ ounces}}{\textit{gallon}}\right) \\ &= 40 \,\frac{\textit{ounces}}{\textit{minute}}\end{aligned}$$

*Similarly, we compute that*

$$\begin{aligned}(\text{rate of salt out}) &= (\text{rate of liquid leaving tank}) \\ &\quad \times (\text{concentration of salt in the tank}) \\ &= \left(\frac{5 \text{ gallons}}{\text{minute}}\right)\left(\frac{S \text{ ounces}}{100 \text{ gallons}}\right) \\ &= \frac{1}{20} S \frac{\text{ounces}}{\text{minute}}\end{aligned}$$

*Thus the differential equation that describes the change in $S$ is*

$$\frac{dS}{dt} = 40 - \frac{1}{20}S. \tag{1.2.7}$$

first-mixing

*Notice that (1.2.7) takes the form of the forced basic growth model (1.2.5) with relative growth rate $r = -1/20$ and forcing $f = 40$.*

**Activity 1.2.3.** *Paul's brother purchases 22 ounce chocolate drink and begins drinking at a rate of 3 ounces per minute. Paul begins to pour strawberry drink, which contains 10% fruit, in to his brother's glass at a rate of 1 ounce per minute. Find a differential equation that models the amount of fruit in the brother's glass as a function of time.*

FirstConstruct

**Exercise 1.2.1.** Construct a differential equation which models the following situations.

1. An investment $y(t)$ grows with relative growth rate of 5% per year;
2. $1000 is invested with annual interest rate of 5%;
3. Continuous deposits are made into an account at the rate of $1000 a year. In addition to these deposits, the account earns 7% interest per year.
4. Paul takes out a loan with an annual interest rate of 6%. Continuous repayments are made totaling $1000 per year.
5. A fish population under ideal conditions grows at a relative growth rate $k$ per year. The carrying capacity of their habitat is $N$ and $H$ fish are harvested each month.
6. A fish population under ideal conditions grows at growth rate $k$ per year. The carrying capacity of their habitat is $N$ and one quarter of the fish population is harvested annually.
7. Due to the pollution problems the relative growth rate of a fish population is a decreasing exponential function of time.

Modeling1

**Exercise 1.2.2.** Some quantity $f$ changes with time, is measured in gallons, and is modeled by the differential equation

$$\frac{df}{dt} = kf\left(1 - \frac{f}{M}\right),$$

where the time variable $t$ is measured in months and the parameter $M$ is also measured in gallons.

1. In what units is $\frac{df}{dt}$ measured?
2. In what units is $1 - \frac{f}{M}$ measured?
3. In what units is $f\left(1 - \frac{f}{M}\right)$ measured?
4. In what units is the parameter $k$ measured?

Modeling2

**Exercise 1.2.3.** A variable quantity $r = r(t)$ is measured in grams per liter and is modeled by the differential equation

$$\frac{dr}{dt} = -\frac{kr}{c+r}.$$

The time variable $t$ is measured in seconds. Figure out the units for the parameters $k$ and $c$.

RW-MidBegin

**Exercise 1.2.4.** We consider a large urn of coffee in cafeteria. A student is draining coffee in to her thermos while at the same time the brewing machine is adding fresh coffee. We want to keep track of the amount of caffeine (in milligrams) in the urn during this process.

Suppose the following:

- There is initially 4 liters of coffee in the urn, with a caffeine concentration of 100 mg per liter.
- Starting at time $t = 0$ the brewing machine is adding extra strong coffee, which has a concentration of 250 mg per liter. This new coffee is being added at a rate of 25 mL per minute.
- Starting at time $t = 0$ the student begins draining coffee out of the urn at a rate of 30 mL per minute.

Write down a differential equation which models the amount of caffeine in the urn as a function of time.

IvaJunkEmail

**Exercise 1.2.5.** In this problem we model the number of junk emails in Iva's inbox with a continuous function $J(t)$. Suppose the following:

- When the semester started ($t = 0$), Iva had 6000 emails in her inbox; 4000 of them were junk.
- Email is continuously flowing in to Iva's email inbox at a rate of 50 per day; 30% of the emails are junk.
- Each day, Iva randomly picks 20 emails to deal with[1] – once they have been dealt with, she moves them out of her inbox.

[1] This is not actually true, of course.

Write down a differential equation describing the number of junk emails in Iva's inbox.

RW-Last

**Exercise 1.2.6.** A big mixing vat contains 50 liters of a mixture in which the concentration of a certain chemical is 1.25 grams per liter. This mixture is being diluted by another mixture in which the concentration of the same chemical is 0.25 grams per liter. Each minute 6 liters of the less concentrated mixture are poured into the vat and 4 liters of the resulting new mixture are drained out. Construct a differential equation modeling this process.