DAY 11

First order systems and parametric curves

ch:first-order-systems

Systems of equations

In this second part of the course we study differential equations that describe two different quantities (which we can think of as populations) whose evolutions are dependent on each other. (The analysis we do in this course can be generalized to systems with more than two quantities. But in this course we restrict attention to situations where there are only two quantities.)

Consider the following examples.

Example 11.1 (Simple competition model). *Suppose that two populations P_1 and P_2 have ideal growth rates r_1 and r_2. Suppose also that the two populations inhabit the same space, and are thus competing for the same habitat. We assume, just as we did for the logistic model, that the relative growth rate for each population is proportional to the available habitat. This assumption leads to the following differential equations*

$$\frac{dP_1}{dt} = r_1 P_1 \left(1 - \frac{P_1 + P_2}{K}\right),$$
$$\frac{dP_2}{dt} = r_2 P_2 \left(1 - \frac{P_1 + P_2}{K}\right),$$

where K is the amount of available habitat. We can re-arrange these equations to take the following form:

$$\frac{dP_1}{dt} = r_1 P_1 \left(1 - \frac{P_1}{K}\right) - \frac{r_1}{K} P_1 P_2,$$
$$\frac{dP_2}{dt} = r_2 P_2 \left(1 - \frac{P_2}{K}\right) - \frac{r_2}{K} P_1 P_2. \tag{11.1}$$

simple-competition-model

We can interpret the first equation in (11.1) by identifying two different terms:

$$\frac{dP_1}{dt} = \underbrace{r_1 P_1 \left(1 - \frac{P_1}{K}\right)}_{L} - \underbrace{\frac{r_1}{K} P_1 P_2}_{I}.$$

The quantity L represents the usual logistic model for population P_1. The quantity I represents the effect that population P_2 has on the growth of population P_1. Note that the quantity I is proportional to both P_1 and P_2.

EXAMPLE 11.2 (Simple predatory-prey model). *Suppose that a population of rabbits, described by $R(t)$, is being hunted by a population of hawks, described by $H(t)$. We assume that in the absence of any hawks, the rabbits grow according to a logistic model. We further assume that the effect of the presence of the hawks causes a negative effect on the size of the rabbit population, and that this effect is proportional to both the number of hawks and the number of rabbits. We furthermore assume that without the presence of rabbits, the hawks would suffer population decline according to a basic growth model, but that the presence of rabbits leads to a positive effect, proportional to both the number of hawks and the number of rabbits.*

The differential equations that describe this situation is

$$\frac{dR}{dt} = r_R R \left(1 - \frac{R}{K}\right) - \alpha RH$$

$$\frac{dH}{dt} = -r_H H + \beta RH,$$

where r_R, r_H, α, and β are all positive constants.

Both of the previous examples lead to a ***system of differential equations***, by which we mean a collection of differential equations for two (or more) functions. Generally, systems are only interesting in situations where the right side of each of the equations depends on more than one function – otherwise, we could just analyze each equation independently. A generic system of differential equations, having unknowns $y_1(t)$ and $y_2(t)$, takes the form

$$\frac{dy_1}{dt} = f_1(y_1, y_2), \qquad \frac{dy_2}{dt} = f_2(y_1, y_2).$$

Here the functions f_1 and f_2 represent the right hand side of the equations.

Visualizing solutions to systems

Before we begin the process of understanding solutions to systems, it is useful to develop several ways to visualize such solutions.

Of course, one way to visualize the two functions is simply make individual plots of each: one plot of y_1 versus t and one plot of y_2 versus t. The problem with having two separate plots is that it is somewhat difficult to understand the impact that y_1 and y_2 have on one another. This motivates us to try to find a way to plot both functions together.

The issue is that we can't draw a two-dimensional plot that has a t axis, a y_1 axis, and also a y_2 axis. The resolution is to simply not include a t axis. Rather, for each time t, we plot the point $(y_1(t), y_2(t))$ on the y_1y_2 plane. As t changes, the values of y_1 and y_2 change accordingly; the result is that we obtain a path in the plane, telling "where" (in y_1y_2 land) we are at each time. The y_1y_2 plane is often called ***state space*** or ***phase space*** or ***configuration space***, because each point in the plane is a possible "state" or "configuration" of the two functions y_1 and y_2. The paths $(y_1(t), y_2(t))$ are a ***parametric curve*** or ***parametric plot*** in state space.

ex:first-parametric

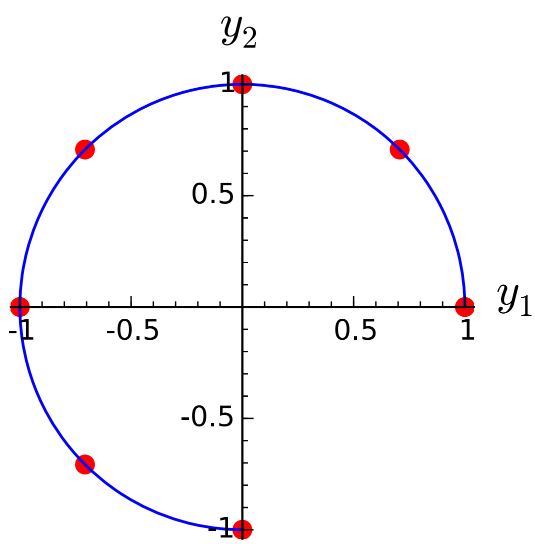
EXAMPLE 11.3. *Suppose*

$$y_1(t) = \cos(t) \qquad y_2(t) = \sin(t).$$

Let's make a plot of several points along the parametric curve that these functions describe.

t	$(y_1(t), y_2(t))$
0	$(\cos(0), \sin(0)) = (1, 0)$
$\pi/4$	$(\cos(\pi/4), \sin(\pi/4)) = (\sqrt{2}/2, \sqrt{2}/2)$
$\pi/2$	$(\cos(\pi/2), \sin(\pi/2)) = (0, 1)$
$3\pi/4$	$(\cos(3\pi/4), \sin(3\pi/4)) = (-\sqrt{2}/2, \sqrt{2}/2)$
	etc.

If we plot these points, we see that the curve is simply traversing the unit circle in a counter clockwise direction:



activity:parametric-1

Activity 11.1. *Make a parametric plot of the functions $y_1(t) = e^t$, $y_2(t) = e^{2t}$. You should find that the resulting curve is a parabola!*

activity:parametric-2

Activity 11.2. *Make a parametric plot of the functions $y_1(t) = 2e^{-t}$, $y_2(t) = 5e^{-t}$. You should find that the resulting curve is a straight line! What happens as $t \to \infty$? What happens as $t \to -\infty$?*

The following example shows that it is relatively straightfoward to have Sage draw parametric plots.

ex:sage-parametric-plot

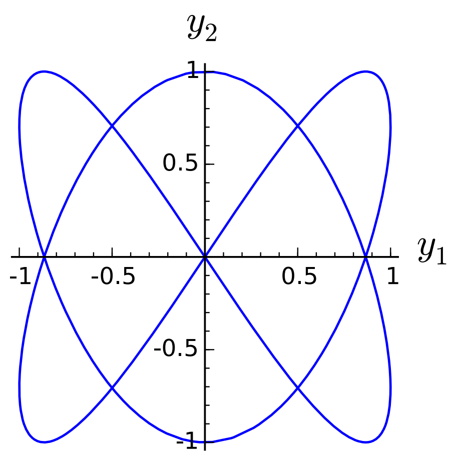
Example 11.4. *Suppose we want Sage to make a parametric plot of the functions*

$$y_1(t) = \sin(2t) \qquad y_2(t) = \sin(3t)$$

for $0 \leq t \leq 2\pi$. We can accomplish this with the following code:

```
var('t')
y1(t) = sin(2*t)
y2(t) = sin(3*t)
parPlot=parametric_plot( (y1(t),y2(t)),(t,0,2*pi),
   axes_labels=['$y_1$','$y_2$'])
parPlot.show(figsize=[4,3])
```

The resulting plot is the following:

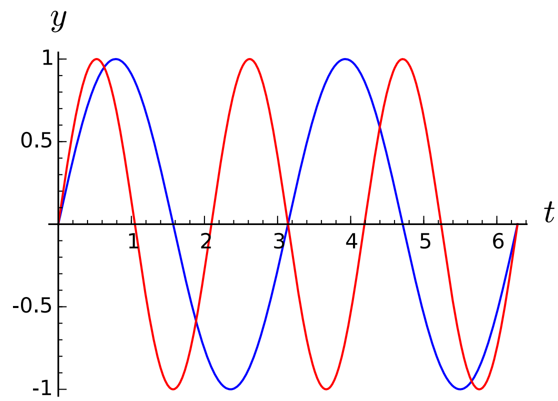


ACTIVITY 11.3. *Have Sage plot the parametric curves defined by the functions in Activity 11.1.*

ACTIVITY 11.4. *Have Sage plot the parametric curves defined by the functions in Activity 11.2.*

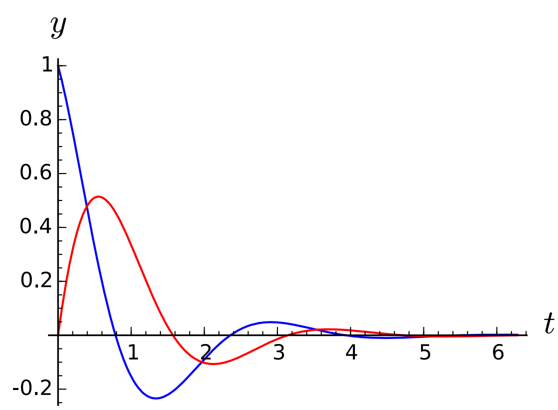
It can be instructive to compare the parametric plot of $y_1(t)$ and $y_2(t)$ with the individual plots of y_1 and y_2. For example, compare the plots in the examples above. If you are given one of the plots, can you reconstruct the other?

ACTIVITY 11.5. *Consider the functions* $y_1(t)$ *and* $y_2(t)$ *whose plots are as follows:*



Here $y_1(t)$ in blue and $y_2(t)$ in red. Based on that figure, construct the parametric plot of these functions.

ACTIVITY 11.6. *Consider the functions $y_1(t)$ and $y_2(t)$ whose plots are as follows:*



where $y_1(t)$ in blue and $y_2(t)$ in red. Based on that figure, construct the parametric plot of these functions.

Exercises

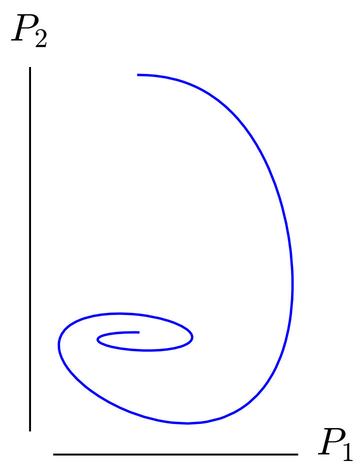
general-competition-model

EXERCISE 11.1. *It may be the case that two populations are competing for the same physical habitat, but that that habitat can sustainably support different numbers of each population. (For example, if chickens and sheep are competing for the grass in a field, the same size field can support very different numbers of chickens and sheep.)*

Construct a generalization of the competition model (11.1) *that takes this in to account by assuming that population* P_1 *has carrying capacity* K_1 *and that population* P_2 *has carrying capacity* K_2 *and defining the "available room" in the habitat by*

$$1 - \frac{P_1}{K_1} - \frac{P_2}{K_2}.$$

EXERCISE 11.2. *Suppose that the populations* P_1 *and* P_2 *follow the following trajectory in phase space.*



Assume that the starting point is at the "top" of the diagram.

(1) *Describe in words what happens to population* P_1.
(2) *Describe in words what happens to population* P_2.

MakeMeAMatch

EXERCISE 11.3. *Match the pictures in Figure* 11.1 *with the parametric equations below. You can use technology to prompt you in the right direction. Make sure that in the end you learn something that can help you do such exercises without any help of "technology".*

(1) $x(t) = \cos(2t)$, $y(t) = \sin(2t)$;
(2) $x(t) = 2\cos(t)$, $y(t) = \sin(t)$;
(3) $x(t) = e^{-t}\cos(t)$, $y(t) = e^{-t}\sin(t)$;
(4) $x(t) = e^{-t}\sin(t)$, $y(t) = e^{-t}\cos(t)$;
(5) $x(t) = (1+\cos^2(5t))\cos(t)$, $y(t) = (1+\cos^2(5t))\sin(t)$;
(6) $x(t) = e^{t}$, $y(t) = -e^{t}$;
(7) $x(t) = \cos(t)$, $y(t) = \cos(t)$;

(8) $x(t) = \cos(t)$, $y(t) = t$;

(9) $x(t) = t$, $y(t) = \cos(2t)$.

DrawCurves

EXERCISE 11.4. *Draw the curves with the following parametric equations; indicate the direction in which t increases. You can use technology to prompt you in the right direction. Make sure that in the end you find a way of making the drawing without any use of "technology".*

(1) $x(t) = \cos\left(\frac{t}{2}\right)$, $y(t) = \sin\left(\frac{t}{2}\right)$;

(2) $x(t) = e^t \cos(t)$, $y(t) = e^t \sin(t)$;

(3) $x(t) = t\cos(t)$, $y(t) = t\sin(t)$;

(4) $x(t) = 2e^t$, $y(t) = -3e^t$;

(5) $x(t) = e^{-t}$, $y(t) = 2e^{-t}$.

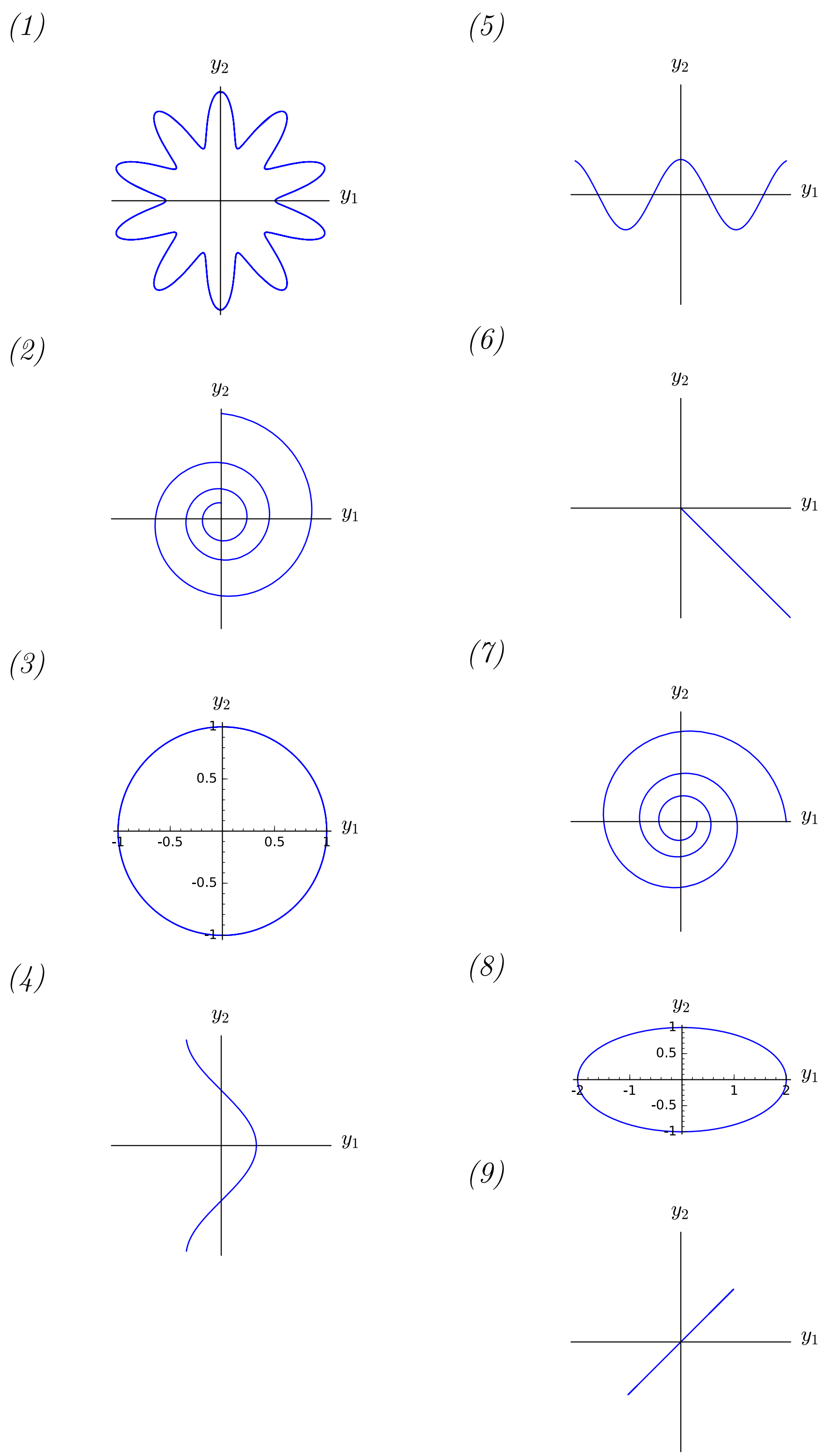


FIGURE 11.1. Parametric curves corresponding to the functions in Exercise 11.3.

MatchParametric