r:what-is-a-differential-equation

1. What is this course about?

A ***differential equation*** is an equation where the unknown quantity is a function, and where the equation involves the derivative(s) of that unknown function. This definition is perhaps a bit abstract, and it's not immediately clear that such objects are useful or interesting. So let's illustrate with an example.

Suppose we want to mathematically describe a population that changes in time. For example, we might want to describe the population of a certain bacteria culture in a jar of yogurt. We can do this with a function $P(t)$ that represents the size of the population at each time t. For the purposes of this example, let's suppose that t is measured in minutes and $P(t)$ is measured in millions of bacteria. The derivative $P'(t)$ tells us the rate of change of the population. It is important to note that the units of $P'(t)$ are the units of P divided by the units of t. A good way to see this is to note that

$$P'(t) = \frac{dP}{dt} = \frac{\text{infinitesimal change in } P}{\text{infinitesimal change in } t}.$$

Thus in our example, $P'(t)$ has units of millions per minute. We might suppose that the rate of change of the population is a constant 7 million per minute. Mathematically, this is described by the relation

$$\frac{dP}{dt} = 7. \tag{1.1}$$

first-ode

The equation (1.1) is a differential equation. The unknown object is P, the function describing the population, and the equation involves the first derivative of P. (Note that we have suppressed the units when writing down the differential equation.)

In fact, (1.1) is an ***ordinary differential equation (ODE)***, which means that the unknown function P depends only on one variable, namely t. ODEs involve only the "ordinary" derivatives studied in the single-variable calculus course. Differential equations where the unknown function depends on multiple variables involve partial derivatives, and are called ***partial differential equations***. In this course we focus primarily on ordinary differential equations.

We make two remarks about (1.1).

- First, the equation (1.1) is a statement about functions, not numbers. It states that the function $P'(t)$ is the same function as the constant function whose output is always 7. This means that (1.1) is to be interpreted as holding at all times t being considered.
- Second, we highlight the process by which we obtained (1.1). We began by describing a physical quantity (the population size) using the function $P(t)$. We then proceeded to make an assumption about the physical quantity – namely, that the rate of change is constant. Subsequently, we translated that assumption in to an equation. This is a standard process for constructing differential equations that describe situations "out in the world."

This course is an introduction to the study of ordinary differential equations. The primary emphasis is on constructing differential equations that describe various biological and physical phenomena, and subsequently studying those differential equations. There are many ways one can "study" a differential equation. First, one would like to know if there are any solutions to the equation; this question is addressed by the Fundamental Theorem of ODEs, discussed in §5, which gives us conditions under which solutions exist. Once it is known that solutions exist, one would like to understand the qualitative behavior of solutions. For example, do the solutions oscillate? Are they bounded? Do they become infinite in finite time? etc.

The course is organized as follows. In Chapter 1 we introduce numerical, qualitative, and quantitative methods for constructing and studying differential equations. In Chapter 2 we study linear and nonlinear first order systems, focusing on equations that describe population dynamics.

Chapter 3 turns to the study second order equations, focusing on equations describing oscillations. Chapter 4 is concerned with physical systems governed by equations arising from the principle of conservation of energy. Finally, in Chapter 5 we explore power series solutions to differential equations, with applications to special functions.

We are assuming only that students have taken a year-long sequence in single-variable calculus. Thus we give short introductions to the necessary material from multivariable calculus and linear algebra.

We believe that the study of differential equations benefits from using technology, especially for visualization purposes. We have included various snippets of code to be used with the open-source product Sage.

We also believe that it is important that students be able to present the results of their work in narrative form. Thus there are writing assignments scattered throughout the notes.

Finally, as is the case with any document this size, there are bound to be errors present in these notes. If you notice an error, please let Paul know! A list of corrections is posted on his website.