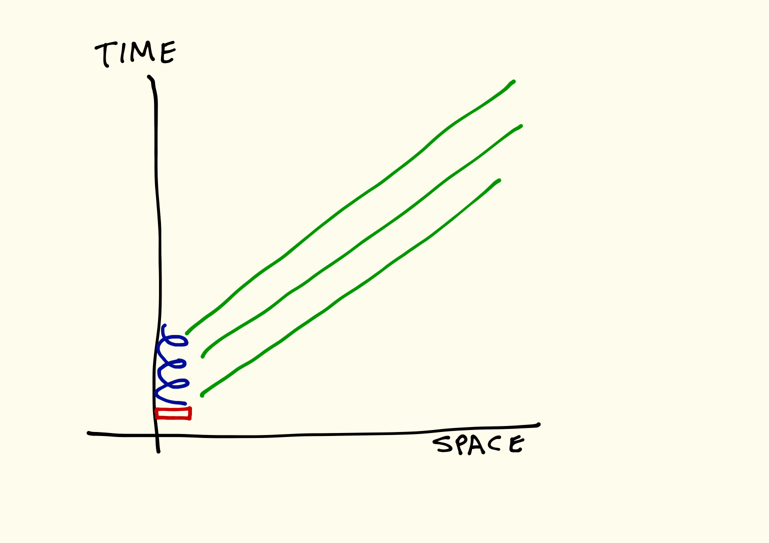
Asymptotic gluing of shear-free hyperboloidal initial data.

Paul T. Allen
Lewis & Clark College

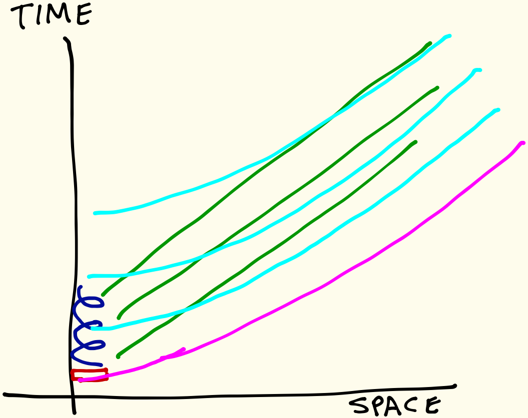
joint work with
James Isenberg, John M. Lee, Iva Stavrov Allen

JMM 2018

Outgoing radiation from a gravitational event

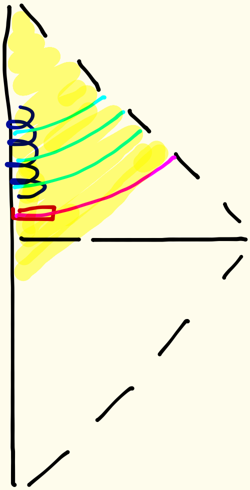


Hyperboloidal foliations



- Geometry of slices is asymptotic to hyperbolic space

Hyperboloidal foliations in compactified spacetime



- Well-suited for future evolution problem
- Compact domains are good for numerics

CMC hyperboloidal data

- Metric $g = \rho^{-2}\overline{g}$ is asymptotically hyperbolic

$$\mathrm{Riem}[g] = -\,\mathrm{id} + \dots \quad \leftrightarrow \quad |d\rho|_{\overline{g}} = 1 + \dots$$

- Constant mean curvature

$$K = -g + \Sigma$$

- Constraint equations

$$\mathrm{R}[g] - |\Sigma|_g^2 + 6 = 0 \qquad \mathrm{div}_g\,\Sigma = 0$$

- Shear-free condition required for compactifiable spacetime

$$\begin{aligned} \Sigma &= \rho^{-1}\left(\mathrm{Hess}_{\overline{g}}\,\rho - \tfrac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g}\right) + \dots \\ &= \rho^{-1}\mathcal{H}_{\overline{g}}(\rho) + \dots \end{aligned}$$

Constructing hyperboloidal data

Andersson-Chruściel-Friedrich, Andersson-Chruściel, Gicquaud-Sakovich, Isenberg-Lee-Stavrov,…

- Start with "seed data": metric λ, tensor μ
- Look for

$$g = \phi^4 \lambda, \quad \Sigma = \phi^{-2}(\mu + \mathcal{D}_\lambda W), \qquad \mathcal{D}_\lambda W = \text{ tracefree } \mathcal{L}_W \lambda$$

- Constraints satisfied if ϕ and W satisfy the elliptic system

$$\mathcal{D}_\lambda^* \mathcal{D}_\lambda W = -\operatorname{div}_\lambda \mu$$

$$\Delta_\lambda \phi = \frac{1}{8} \mathrm{R}[\lambda]\phi - \frac{1}{8}|\mu + \mathcal{D}_\lambda W|_g^2 \phi^{-7} + \frac{3}{4}\phi^5$$

- Appropriate Hölder theory available

Constructing *shear-free* hyperboloidal data

- To get boundary regularity use intermediate spaces

$$C^{k,\alpha}(\overline{M}) \subset \mathscr{C}^{k,\alpha;m}(M) \subset C^{k,\alpha}(M)$$

- Seed metric $\lambda = \rho^{-2}\overline{\lambda}$ with

$$\overline{\lambda} \in \mathscr{C}^{k,\alpha;2}(M) \subset C^{1,1}(\overline{M})$$

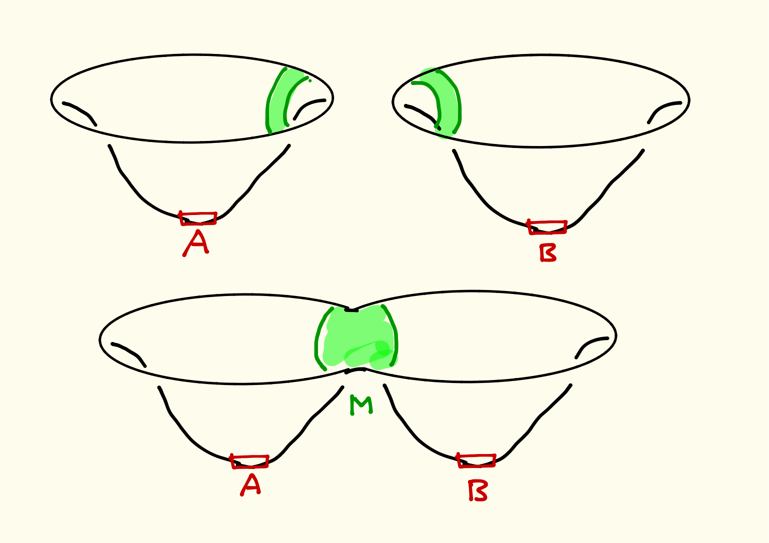
- Seed tensor

$$\mu = \rho^{-1}\mathcal{H}_{\overline{\lambda}}(\rho) + \dots$$

- Solve for $\phi = 1 + O(\rho^2)$ to get

$$\overline{g} = \phi^4\overline{\lambda} \in \mathscr{C}^{k,\alpha;2}(M)$$

Constructing data for two-body problems I



Isenberg-Lee-Stavrov gluing theorem

- For $0 < \epsilon \ll 1$ form connect sum M_ϵ
- Construct seed data λ_ϵ, K_ϵ
- Apply conformal method for each ϵ
- Uniform estimates give convergence in exterior region

Friendly (retrospective) critique

- Shear-free condition missing
- Convergence in "physical topology"
- What's happening in middle region?

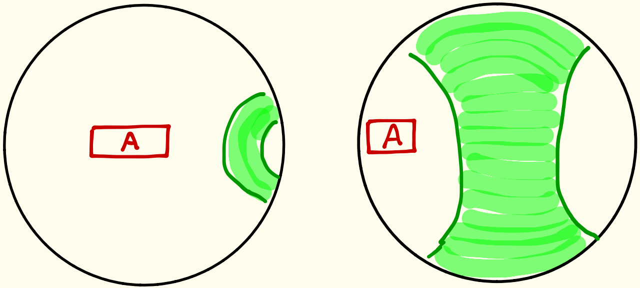
A-Stavrov density theorem

- (g, K) "polyhomogeneous" data, not necessarily shear-free
- Construct shear-free $(g_\epsilon, K_\epsilon) \to (g, K)$ in physical topology

Applications

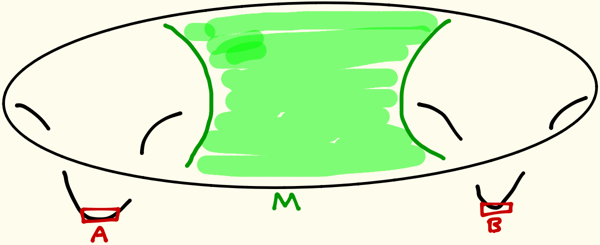
- Shear-free data is "sufficiently general"?
- Stronger topology needed in convergence results

Geometry of the gluing region



- Data in gluing region approximates a slice of Minkowski spacetime

Improved gluing theorem



- $(M_\epsilon, g_\epsilon, K_\epsilon)$ are shear-free
- Exterior regions converge strongly to original data
- Middle region converges strongly to Minkowski hyperboloid

A peak under the rug

Need estimates, *uniform in* ϵ, in each region:

- Construct approximate solution $\mathcal{N}_\epsilon(\phi_\epsilon^{\text{approx}}) \approx 0$
- Linearize about approximate solution $\phi_\epsilon = \phi_\epsilon^{\text{approx}} + u_\epsilon$

$$0 = \mathcal{N}_\epsilon(\phi_\epsilon^{\text{approx}} + u_\epsilon) = \mathcal{N}_\epsilon(\phi_\epsilon^{\text{approx}}) + \mathcal{L}_\epsilon u_\epsilon + \mathcal{Q}_\epsilon(u_\epsilon)$$

- Blowup analysis: uniform estimates for linearized operators
- Solve fixed-point problem in ϵ-ball

$$u_\epsilon = \mathcal{L}_\epsilon^{-1}\left(\mathcal{N}_\epsilon(\phi_\epsilon^{\text{approx}}) + \mathcal{Q}_\epsilon(u_\epsilon)\right)$$

- Function spaces with weights adapted to the gluing