3.3 Differential equations

Definition 3.38. *Let $U \subset \mathbb{R}$ be open, $x_0 \in U$, and $F\colon U \to \mathbb{R}$ be a continuous function.*

1. *The **initial value problem** determined by x_0 and F seeks to find $T > 0$ and function $x\colon (-T, T) \to \mathbb{R}$ such that*

$$x' = F \circ x \qquad \textit{and} \qquad x(0) = x_0.$$

*A **solution** to the initial value problem determined by x_0 and F is a function $x \in C^1((-T, T))$ satisfying these conditions.*

2. *The **integral equation** determined by x_0 and F seeks to find $T > 0$ and function $x\colon (-T, T) \to \mathbb{R}$ such that*

$$x(t) = x_0 + \int_0^t (F \circ x)(s)\, ds.$$

*A **solution** to the integral equation determined by x_0 and F is a function $x \in C^0((-T, T))$ satisfying this condition.*

Lemma 3.39. *Let $U \subset \mathbb{R}$ be open, $x_0 \in U$, and $F\colon U \to \mathbb{R}$ be a continuous function. A function $x\colon (-T, T) \to \mathbb{R}$ is a solution to the initial value problem determined by x_0 and F if and only if it is a solution to the integral equation determined by x_0 and F.*

Proof. Suppose x is a solution to the initial value problem and let $t \in (-T, T)$. Without loss of generality, $t > 0$. Applying the fundamental theorem of calculus to x' on the interval $[0, t]$ yields

$$x(t) - x(0) = \int_0^t (F \circ x)(s)\, ds,$$

which immediately implies that x satisfies the integral equation.

Suppose instead that x satisfies the integral equation. Clearly we have $x(0) = 0$. The fundamental theorem of calculus implies that

$$x' = F \circ x.$$

Since both F and x are continuous, this implies that x' is continuous. Thus $x \in C^1(*-T, T))$ and is therefore a solution to the initial value problem. □

Theorem 3.40 (Peano existence theorem). *Let $U \subset \mathbb{R}$ be open, $x_0 \in U$, and $F \colon U \to \mathbb{R}$ be a continuous function. Then there exists at least one solution to the initial value problem determined by x_0 and F.*

Proof. Fix $\Delta > 0$ such that $\overline{B_\Delta(x_0)} \subset U$ and let

$$M = \sup_{B_\Delta(x_0)} |F|.$$

For each $n \in \mathbb{N}$ we define the sequence $x_n^{-n}, \dots, x_n^{-1}, x_n^0, x_n^1, \dots, x_n^n$ by

$$\begin{aligned} x_n^0 &= x_0, \\ x_n^k &= x_n^{k-1} + \frac{\Delta}{Mn} F(x_n^{k-1}), \quad k > 0, \\ x_n^k &= x_n^{k+1} - \frac{\Delta}{Mn} F(x_n^{k+1}), \quad k < 0 \end{aligned}$$

Claim 1: Each sequence $\{x_n^k\}_{k=-n}^n$ is contained in $\overline{B_\Delta(x_0)}$.

Let $T = \Delta/M$. For each $n \in \mathbb{N}$ we define the function $x_n \colon [-T, T] \to \mathbb{R}$ by requiring that for each $k = 0, \dots, n-1$ we have

$$x_n(t) = x_n^k + F(x_n^k)\left(t - \frac{kT}{n}\right), \quad \text{when} \quad \frac{kT}{n} \le t \le \frac{(k+1)T}{n}$$

and that for each $k = -n, \dots, -1$ we have

$$x_n(t) = x_n^k + F(x_n^{k+1})\left(t - \frac{kT}{n}\right), \quad \text{when} \quad \frac{kT}{n} \le t \le \frac{(k+1)T}{n}$$

Claim 2: We claim that for each $n \in \mathbb{N}$ the function $x_n \colon [-T, T] \to \mathbb{R}$ is continuous.

Claim 3: We claim that for each $n \in \mathbb{N}$ and for each $t, s \in [-T, T]$ we have

$$|x_n(t) - x_n(s)| \le M|t - s|.$$

Claim 4: We claim that the sequence $\{x_n\}_{n=1}^\infty \subset C^0([-T, T])$ is uniformly bounded and equicontinuous.

Once Claim 4 is established, we can invoke the Arzela-Ascoli theorem to conclude that there exists some subsequence $\{x_{n_k}\}_{k=1}^\infty$ that converges to function $x \in C^0([-T, T])$.

Claim 5: The function x is a solution to the integral equation determined by x_0 and F. $\square$

Lemma 3.41. *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and is differentiable on (a,b). If there exists $M > 0$ such that*

$$|f'| \leq M|f| \qquad \text{on } (a,b)$$

and there exists $x \in [a,b]$ such that $f(x) = 0$, then f is identically zero.

Proof. Let $Z = \{x \in [a,b] \mid f(x) = 0\}$. By hypothesis, Z is nonempty. Furthermore, since Z is the continuous preimage of the closed set $\{0\}$, we sse that Z is closed. We claim that Z is also open.

Suppose $x \in Z$. Let $F = \sup_{\overline{B_{1/2M}(x)}} |f|$. For any $y \in \overline{B_{1/2M}(x)} \setminus \{x\}$ we have

$$\begin{aligned} |f(y)| &= |f(y) - f(x)| \\ &= \left| \int_x^y f'(t)\, dt \right| \\ &\leq \int_x^y |f'(t)|\, dt \\ &\leq \int_x^y M|f(t)|\, dt \\ &\leq \int_x^y MF\, dt \\ &= MF|x - y| \\ &\leq F/2. \end{aligned}$$

This implies that for all $y \in \overline{B_{1/2M}(x)}$ we have

$$|f(y)| \leq \frac{1}{2} \left(\sup_{\overline{B_{1/2M}(x)}} |f| \right),$$

which is only possible if $f = 0$ on $\overline{B_{1/2M}(x)}$. Thus $B_{1/2M}(x) \subset Z$ and Z is open.

Since $[a,b]$ is connected, and since Z is nonempty and both open and closed, we must have $Z = [a,b]$. Thus f is the zero function. $\square$

Definition 3.42. *Let $U \subset \mathbb{R}$. A function $f\colon U \to \mathbb{R}$ is* ***Lipschitz continuous*** *if there exists constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in U$.*

Theorem 3.43 (Picard existence and uniqueness theorem)**.** *Let $U \subset \mathbb{R}$ be open, $x_0 \in U$, and $F \colon U \to \mathbb{R}$ be a Lipschitz continuous function. Then there exists $T > 0$ such that the initial value problem determined by x_0 and F has a unique solution in $C^1((-T, T))$.*

Proof. Let M be a Lipschitz constant for function F and let $\Delta > 0$ such that $\overline{B_\Delta(x_0)} \subset U$.

Consider $T > 0$, to be chosen below, depending only on M, Δ, and $F(x_0)$. Define the mapping

$$\mathscr{F} \colon C^0([-T, T]; B_\Delta(x_0)) \to C^0([-T, T]; \mathbb{R})$$

by

$$\mathscr{F}(x)(t) = x_0 + \int_0^t F(x(s))\, ds.$$

Note that for $t \in [-T, T]$ we have

$$|\mathscr{F}(x)(t) - x_0| \le \int_0^T (|F(x(s)) - F(x_0)| + |F(x_0)|)\, ds \le T\,(M\Delta + F(x_0))\,.$$

We choose $T > 0$ to satisfy

$$T < \frac{\Delta}{2\,(M\Delta + F(x_0))}.$$

Thus $\mathscr{F}$ is in fact a map $C^0([-T, T]; B_\Delta(x_0)) \to C^0([-T, T]; B_\Delta(x_0))$.

Let $x, y \in C^0([-T, T]; B_\Delta(x_0))$. We compute

$$\|\mathscr{F}(x) - \mathscr{F}(y)\|_{L^\infty} \le \int_0^T |F(x(s)) - F(y(s))|\, ds \le MT\|x - y\|_{L^\infty}.$$

Thus if we furthermore choose $T > 0$ to satisfy

$$T < \frac{1}{M}$$

then $\mathscr{F}$ is a contraction.

From the Banach fixed-point theorem there exists a unique function $x \in C^0([-T, T]; B_\Delta(x_0))$ such that $\mathscr{F}(x) = x$. Such a function is a solution to the integral equation determined by x_0 and F, and thus is the desired unique solution. □

3.4 The exponential function

Theorem 3.44 (Exponential function)**.** *There exists a unique bijection* $\exp\colon \mathbb{R} \to (0, \infty)$ *such that*

$$\exp' = \exp \qquad \textit{and} \qquad \exp(0) = 1.$$

Furthermore,

1. $\exp(x) > \exp(y)$ *whenever* $x > y$*,*
2. $\exp(x + y) = \exp(x)\exp(y)$ *for all* $x, y \in \mathbb{R}$*, and*
3. $\exp(x)^q = \exp(qx)$ *for all* $x \in \mathbb{R}$ *and* $q \in \mathbb{Q}$*.*

Corollary 3.45 (Logarithm function)**.** *The bijection* $\log\colon (0, \infty) \to \mathbb{R}$ *defined by* $\log = \exp^{-1}$ *satisfies the following properties:*

1. $\log'(x) = \frac{1}{x}$*,*
2. $\log(xy) = \log(x) + \log(y)$ *for all* $x, y > 0$*, and*
3. $\log(x^q) = q \log(x)$ *for all* $x > 0$ *and* $q \in \mathbb{Q}$*.*

Corollary 3.46. *Let* $a \in (0, \infty)$*. The function* $f : \mathbb{Q} \to \mathbb{R}$ *given by* $f(p) = a^p$ *extends to a differentiable function* $f : \mathbb{R} \to \mathbb{R}$*.*

3.5 Approximation theorems

Definition 3.47. *We denote the collection of all polynomials by* $\mathscr{P}$*. We denote the collection of all multivariable polynomials on* $\mathbb{R}^n$ *by* $\mathscr{P}_n$*.*

Definition 3.48. *Suppose* $[a, b] \subset \mathbb{R}$*. A sequence of functions* $\{q_n\}_{k=1}^{\infty} \subset C^0(\mathbb{R})$ *is an* ***approximate identity for*** $[\boldsymbol{a}, \boldsymbol{b}]$ *if*

1. *for each* $n \in \mathbb{N}$ *we have* $\int_a^b q_n(x)\, dx = 1$*, and*
2. *for each* $\delta > 0$ *we have* $q_n \to 0$ *uniformly on* $[a, b] \setminus B_\delta(0)$*.*

Lemma 3.49. *There exists an approximate identity for* $[-1, 1] \subset \mathbb{R}$ *consisting of polynomials.*

Proof. For each $n \in \mathbb{N}$, let $q_n(x) = a_n(1 - x^2)^n$, where the constants a_n are chosen so that $\int_{-1}^{1} q_n(x)\, dx = 1$.

We now address the uniform convergence property; it suffices to consider $\delta \in (0, 1]$. In order to see that $q_n \to 0$ uniformly on $[-1, 1] \setminus B_\delta(0)$, we first observe that Bernoulli's inequality implies $q_n(x) \geq a_n(1 - nx^2)$ for $x \in [-1, 1]$. Integrating this inequality, we deduce that $a_n \leq 1/\sqrt{n}$. This implies that for $1 \geq |x| \geq \delta$ we have

$$|q_n(x)| \leq \frac{(1 - \delta^2)^n}{\sqrt{n}},$$

which clearly tends to zero as $n \to \infty$ at a rate independent of x. □

Theorem 3.50 (Weierstrass approximation theorem)**.** *Consider the closed interval* $[a, b] \subset \mathbb{R}$*. The set of polynomials* $\mathscr{P}$ *is dense in* $C^0([a, b])$*.*

Proof. We first note that, since there is a linear bijection $[a, b] \to [0, 1]$, it suffices to consider the case when $[a, b] = [0, 1]$. Observe also that, because

$$g(x) = f(x) - f(0) - x\,(f(1) - f(0))$$

is a polynomial if and only if $f(x)$ is, it suffices to show that any function $f \in C^0([0, 1])$ such that $f(0) = 0 = f(1)$ can be approximated by polynomials.

Suppose, therefore, that $f \in C^0([0, 1])$ such that $f(0) = 0 = f(1)$. We extend f to a function in $C^0(\mathbb{R})$ by defining $f(x) = 0$ if $x \notin [0, 1]$. Let $\{q_n\}_{n=1}^{\infty}$ be an approximate identity for $[-1, 1]$ consisting of polynomials and consider the polynomial functions

$$p_n(x) = \int_0^1 f(t)\, q_n(t - x)\, dt.$$

By changing variables, we see that

$$p_n(x) = \int_{0-x}^{1-x} f(y + x)\, q_n(y)\, dy = \int_{-1}^{1} f(y + x)\, q_n(y)\, dy.$$

Since $[0, 1]$ is compact, the function f is uniformly continuous. Thus for any $\varepsilon > 0$ we may choose $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/2$. Furthermore, f is bounded with $\|f\|_{L^\infty([0,1])} \leq M$ for some $M > 0$. We subsequently choose $N \in \mathbb{N}$ such that $n \geq N$ implies that $8M\, \|q_n\|_{L^\infty([-1,1] \setminus B_\delta(0))} < \varepsilon$.

For $n \geq N$ the properties of q_n imply that

$$\begin{aligned}
|p_n(x) - f(x)| &= \left| \int_{-1}^{1} (f(y+x) - f(x)) q_n(y)\, dy \right| \\
&\leq \int_{-1}^{\delta} 2M q_n(y)\, dy + \int_{-\delta}^{\delta} \frac{\varepsilon}{2} q_n(y)\, dy + \int_{\delta}^{1} 2M q_n(y)\, dy \\
&\leq 4M \|q_n\|_{L^\infty([-1,1]\setminus B_\delta(0))} + \frac{\varepsilon}{2} \int_{-1}^{1} q_n(y)\, dy \\
&< \varepsilon.
\end{aligned}$$

From this we conclude that $n \geq N$ implies that $\|p_n - f\|_{L^\infty([0,1])} < \varepsilon$, which implies the desired result. □

Remark 3.51 (Stone-Weierstrass theorem)**.** *The Weierstrass approximation theorem can be greatly generalized. Let K be a compact metric space. A set $\mathscr{A} \subset C^0(K)$ is called a **unital algebra** if*

- *the constant function 1 is in $\mathscr{A}$,*
- *whenever $f, g \in \mathscr{A}$ we have both $f + g \in \mathscr{A}$ and $f \cdot g \mathscr{A}$, and*
- *whenever $f \in \mathscr{A}$ and $c \in \mathbb{R}$ we have $c \cdot f \in \mathscr{A}$.*

Suppose that $\mathscr{A} \subset C^0(K)$ is a unital algebra.

1. *We say that $\mathscr{A}$ is **nowhere vanishing** if for each $\mathbf{x} \in K$ there exists $f \in \mathscr{A}$ such that $f(\mathbf{x}) \neq 0$.*
2. *We say that $\mathscr{A}$ **separates K** if for all distinct $\mathbf{x}, \mathbf{y} \in K$ there exists $f \in \mathcal{A}$ such that $f(\mathbf{x}) \neq f(\mathbf{y})$.*

The Stone-Weierstrass theorem states that if K is a compact metric space and if $\mathscr{A} \subset C^0(K)$ is a nowhere vanishing unital algebra that separates K then $\mathscr{A}$ is dense in $C^0(K)$. A proof of the theorem appears in Principles of Mathematical Analysis *by Walter Rudin.*

An immediate consequence of the Stone-Weierstrass theorem is that if $K \subset \mathbb{R}^n$ is compact then $\mathscr{P}_n$ is dense in $C^0(K)$.